On the structure of proper holomorphic mappings

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Declaration

I hereby declare that the work reported in this thesis is entirely original and has been carried out by me under the supervision of Prof. Gautam Bharali at the Department of Mathematics, Indian Institute of Science, Bangalore. I further declare that this work has not been the basis for the award of any degree, diploma, fellowship, associateship or similar title of any University or Institution.

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Abstract

The aim of this dissertation is to give explicit descriptions of the set of proper holomorphic mappings between two complex manifolds with reasonable restrictions on the domain and target spaces. Without any restrictions, this problem is intractable even when posed for domains in $\mathbb{C}^n$. We give partial results for special classes of manifolds. We study, broadly, two types of structure results:

Descriptive. The first result of this thesis is a structure theorem for finite proper holomorphic mappings between products of connected, hyperbolic open subsets of compact Riemann surfaces. A special case of our result follows from the techniques used in a classical result due to Remmert and Stein, adapted to the above setting. However, the presence of factors that have no boundary or boundaries that consist of a discrete set of points necessitates the use of techniques that are quite divergent from those used by Remmert and Stein. We make use of a finiteness theorem of Imayoshi to deal with these factors.

Rigidity. A famous theorem of H. Alexander proves the non-existence of non-injective proper holomorphic self-maps of the unit ball in $\mathbb{C}^n$, $n > 1$. Several extensions of this result for various classes of domains have been established since the appearance of Alexander's result, and it is conjectured that the result is true for all bounded domains in $\mathbb{C}^n$, $n > 1$, whose boundary is C^2-smooth. This conjecture is still very far from being settled. Our first rigidity result establishes the non-existence of non-injective proper holomorphic self-maps of bounded, balanced pseudoconvex domains of finite type (in the sense of D'Angelo) in $\mathbb{C}^n$, $n > 1$. This generalizes a result in $\mathbb{C}^2$, by Coupet, Pan and Sukhov, to higher dimensions. As in Coupet–Pan–Sukhov, the aforementioned domains *need not* have real-analytic boundaries. However, in higher dimensions, several aspects of their argument do not work. Instead, we exploit the circular symmetry and a recent result in complex dynamics by Opshtein.

Our next rigidity result is for bounded symmetric domains. We prove that a proper holomorphic map between two non-planar bounded symmetric domains of the same dimension, one of them being irreducible, is a biholomorphism. Our methods allow us to give a single, all-encompassing argument that unifies the various special cases in which this result is known. Furthermore, our proof of this result *does not* rely on the fine structure (in the sense of Wolf et al.) of bounded symmetric domains. Thus, we are able to apply our techniques to more general classes of domains. We illustrate this by proving a rigidity result for certain convex balanced domains whose automorphism groups are assumed to only be non-compact. For bounded symmetric domains, our key tool is that of Jordan triple systems, which is used to describe the boundary geometry.

Contents

1 Introduction

In this dissertation, we present some results that, broadly speaking, fall into the category of structure results for proper holomorphic mappings. Recall that a continuous mapping $f : X \to Y$ between topological spaces X and Y is said to be *proper* if $f^{-1}(K)$ is compact whenever $K \subset Y$ is compact. A basic example: the proper holomorphic self-mappings of the unit disk $\mathbb{D}$ are precisely the finite Blaschke products. Proper holomorphic mappings are interesting from a number of perspectives. For example:

1. **From the perspective of being holomorphic.** Holomorphicity confers such rigidity on a proper holomorphic map that, in many circumstances, it is not much unlike a biholomorphism (see Section 2.1, where we give a summary of the basic properties of proper holomorphic mappings). In fact, one of the central themes of this dissertation is motivated by the conjectured meta-principle that if a domain is sufficiently "nice", then the set of proper holomorphic self-maps and the set of automorphisms of the domain coincide.

2. **From the perspective of geometry.** Compact manifolds are, in general, easier to study than non-compact ones. Loosely speaking, the number of holomorphic maps between pairs of complex manifolds is — in general — small. This principle has been established in diverse ways in the literature, with the compactness of the underlying manifolds playing a crucial part. No such principles exist for the full class of holomorphic maps between *non-compact* manifolds. When studying non-compact manifolds (like domains in $\mathbb{C}^n$), the condition of properness often serves as a substitute for compactness. For instance, proper holomorphic mappings between domains are finite mappings (i.e., the inverse image of a point is a finite set), which is reminiscent of the behaviour at regular values of the inverse images of smooth mappings of compact manifolds of the same dimension; see Result 2.1.2.

A precise description (such as what we have for the unit disk) for the set of all proper holomorphic mappings between two complex manifolds is, in general, very hard to obtain. The purpose of our work has been to derive such descriptions for certain classes of domains/manifolds. We present two types of structure results for proper holomorphic mappings. The first deals with manifolds that are products of certain Riemann surfaces. We give the structure of a proper holomorphic between two such manifolds in terms of the proper holomorphic mappings between the individual factors of the two manifolds. The second type of results are what may be called rigidity results. We show that for certain reasonably "nice" domains, the class of proper holomorphic mappings and biholomorphisms coincide.

We shall have a discussion of these results in Section 1.5. Before this, however, we shall briefly survey some of the important results in the literature that are connected with these two themes.

1.1 Early results

As we had remarked in the previous section, it is, in general, very hard to give a description of the set of all proper holomorphic maps between two given complex manifolds. We have such a description for proper holomorphic self-maps of the unit disk $\mathbb{D} \subset \mathbb{C}$. The set of all proper holomorphic self-maps of $\mathbb{D}$ are precisely the finite Blaschke products. One of the simplest classes of domains in $\mathbb{C}^n$ is the class of products of planar domains. It is, therefore, natural to ask whether one can obtain a precise description of the proper holomorphic self-maps of the unit polydisk $\mathbb{D}^n$, $n > 1$. The following theorem is a consequence of Cartan's uniqueness theorem, and gives the precise description of the automorphisms of $\mathbb{D}^n$.

Theorem 1.1.1. *For each $f := (f_1, \ldots, f_n) \in \mathrm{Aut}(\mathbb{D}^n)$, there exists a permutation p of $\{1, \ldots, n\}$ such that each f_j, $j = 1, \ldots, n$, is of the form $f_j(z_{p(j)})$. Consequently, each $f_j : \mathbb{D} \to \mathbb{D}$ is an automorphism.*

The above result gives some hope of obtaining a structure result for proper holomorphic mappings between products of planar domains. The following theorem due to Remmert and Stein gives the precise structure of a proper holomorphic map between certain product domains.

Theorem 1.1.2 (Remmert and Stein [RS60])**.** *Let $D = D_1 \times D_2 \times \cdots \times D_n$ and $W = W_1 \times W_2 \times \cdots \times W_n$ be such that each $D_j \subsetneq \mathbb{C}$ is a domain with $\mathbb{C} \setminus D_j$ having non-empty interior, and each $W_j \subsetneq \mathbb{C}$ is a bounded domain. Let $f = (f_1, \ldots, f_n)$ be a proper holomorphic map from D to W. Then, each f_j, $j = 1, \ldots, n$, is of the form $f_j(z_{p(j)})$, where p is a permutation of $\{1, \ldots, n\}$.*

Remark 1.1.3. The proof of this result in the case $n = 2$ was given by Remmert and Stein ([RS60, Satz 12]). Their proof uses Rado's theorem. The proof of the general case requires a generalization of Rado's theorem, but all other aspects of Remmert and Stein's proof remain unchanged. For a proof of the above result and other related results, refer to [Nar71, pp. 71–78].

Another result on the theme of mappings between product spaces is the following theorem of Peters which generalizes a well-known result by Cartan [Car74]:

Theorem 1.1.4 (Peters [Pet74]). *Let X and Y be hyperbolic complex spaces. Then the natural injection* $\mathrm{Aut}(X) \times \mathrm{Aut}(Y) \to \mathrm{Aut}(X \times Y)$ *induces an isomorphism*

$$\mathrm{Aut}^0(X) \times \mathrm{Aut}^0(Y) \cong \mathrm{Aut}^0(X \times Y).$$

Here $\mathrm{Aut}^0(X)$ denotes the connected component of the identity element of $\mathrm{Aut}(X)$.

To the best of our knowledge, there is no analogue of the above result for proper holomorphic maps in the literature, except for Theorem 1.1.2 (and a small technical improvement thereof in [Nar71, p. 77]). One would have expected a similar result with the planar domains of Theorem 1.1.2 replaced by hyperbolic Riemann surfaces. However, it is far from clear whether the methods seen in the proofs of *either* of the above results are alone decisive in proving the hoped-for generalization. We investigated the role of (Kobayashi) hyperbolicity in the Remmert–Stein theorem and discovered that the key ingredient that is needed to extend the Remmert–Stein theorem to hyperbolic Riemann surfaces is the finiteness theorem of Imayoshi [Ima83]. A precise statement and a brief discussion of our extension of the Remmert–Stein theorem is given in Section 1.5.

1.2 Boundary regularity of proper holomorphic mappings

Before we can present a survey of results related to our second theme (i.e., rigidity results), we must first survey some results on the boundary regularity of proper holomorphic mappings. The reason for this will become apparent in Section 1.3, but we mention here that almost all known rigidity results (including ours) rely crucially on the extension of the proper holomorphic mapping under consideration up to or beyond the boundary. Our survey will be brief. The literature on this subject is truly enormous. This brief survey relies upon a part of the very comprehensive survey [For93].

Throughout this chapter, whenever we use use the word "smooth", it will refer to C^∞-smoothness unless specified otherwise.

The following theorem of Painlevé is one of the earliest results on boundary regularity.

Theorem 1.2.1 (Painlevé). *Any biholomorphic map of a bounded simply connected domain with smooth boundary onto the unit disk extends smoothly up to the boundary.*

Almost a 100 years after Painlevé's result, Fefferman proved the following theorem which is a far-reaching extension of Theorem 1.2.1.

Theorem 1.2.2 (Fefferman [Fef74]). *Every biholomorphic mapping between two bounded, strictly pseudoconvex domains with smooth boundaries in $\mathbb{C}^n$ extends to a smooth diffeomorphism of their closures.*

Fefferman's theorem stimulated intense activity on the problem of smooth extension to the boundary of biholomorphisms (and later proper holomorphic mappings) between bounded domains with C^∞-smooth boundaries. From the very beginning, there were efforts to simplify Fefferman's proof in such a way that ways to extend his result to weakly pseudoconvex domains would become evident. For instance, it was found by Webster [Web79] that extendability of biholomorphic mappings would follow from a few crucial properties of the Bergman kernel. A little later, a precise sufficient condition, known as Condition R, was formulated by Bell and Ligocka [BL80]. In what follows, $D \subset \mathbb{C}^n$ is a bounded domain, and $B(D) := \mathcal{O}(D) \cap L^2(D)$ is the space of holomorphic functions on D that are square-integrable with respect to the Lebesgue measure, i.e., the Bergman space of D. It is an elementary fact that $B(D)$ is a closed subspace of $L^2(D)$. The orthogonal projection onto $B(D)$, $P : L^2(D) \to B(D)$, is known as the *Bergman projection*. In the terminology of Bell–Ligocka, D is said to satisfy *Condition R* if P is globally regular, i.e., P maps the subspace $C^\infty(\overline{D}) \subset L^2(D)$ into itself.

We now state the generalization of Fefferman's theorem given by Bell and Ligocka.

Theorem 1.2.3 (Bell and Ligocka [BL80]). *Let D and D' be bounded domains with smooth boundaries in $\mathbb{C}^n$ that satisfy Condition R. Then every biholomorphic map of D onto D' extends smoothly to the closure of D.*

The methods deployed by Bell–Ligocka, apart from being substantially simpler than the methods of Fefferman, have the feature that many aspects can also be applied to proper holomorphic mappings. This motivated a very large number of results, including results on local regularity and C-R regularity. Since this aspect of the study of proper holomorphic maps is the most distant from our work — and the associated literature is truly enormous — we shall only survey some of the highlights.

The following result by Bell and Catlin, and independently by Diederich and Fornæss, is the analogue of Theorem 1.2.3 for proper holomorphic mappings.

Theorem 1.2.4 ([Bell–Catlin [BC82], Diederich–Fornæss [DF82]). *Let D and D' be bounded pseudoconvex domains with smooth boundaries in $\mathbb{C}^n$, and let D satisfy Condition R. Then every proper holomorphic map of D onto D' extends smoothly to the closure of D.*

As in Theorem 1.2.3, the pseudoconvexity hypothesis in the above theorem can be dropped if both D and D' are assumed to satisfy Condition R [Bel84b].

The above results prompted the question of whether every bounded domain with smooth boundary satisfies Condition R. This was shown to be false by Barrett [Bar84]. The most general method available to verify that a given domain satisfies Condition R involves checking whether the domain admits a *subelliptic estimate* at every boundary point (see [DK99] for details). The existence of subelliptic estimates at every boundary point of a domain is a sufficient condition for the domain to satisfy Condition R, but it is *not* a necessary condition. The following result of Catlin gives a necessary and sufficient condition for a smoothly bounded pseudoconvex domain to have a subelliptic estimate at a given boundary point.

Theorem 1.2.5 (Catlin [Cat87]). *Let $D \subset \mathbb{C}^n$ be a bounded pseudoconvex domain with smooth boundary. There is a subelliptic estimate at a boundary point $p \in \partial D$ if and only if p is a point of (D'Angelo) finite type.*

We shall define rigorously the notion of finite type, in the sense of D'Angelo, in Chapter 4.

In particular, the above the theorem shows that all the extension results so far described are applicable to bounded pseudoconvex domains with smooth boundaries that are also of finite type. On the other hand, the following result by Boas and Straube shows that there is a natural class of domains, whose boundaries need *not* be of finite type, that also satisfy Condition R.

Theorem 1.2.6 (Boas and Straube [BS91]). *Let $D \subset \mathbb{C}^n$ be a bounded pseudoconvex domain with smooth boundary that admits a smooth defining function that is plurisubharmonic at the boundary of D. Then D satisfies Condition R. In particular, any convex domain with smooth boundary satisfies Condition R.*

One can also exploit symmetries to construct many examples of domains that satisfy Condition R but *do not* admit subelliptic estimates at all boundary points. The most general symmetry condition that one can impose in order to ensure that Condition R is satisfied is to require the domain to have transverse symmetries.

Definition 1.2.7. Let $D \subset \mathbb{C}^n$ be a bounded domain with smooth boundary. We say that D has *transverse symmetries* if there exists a Lie subgroup $G \subset \mathrm{Aut}(D)$ (note that, as D is bounded, $\mathrm{Aut}(D)$ is a (real) Lie group by a classical result of Cartan) such that the natural action $\psi : G \times D \to D, (g, z) \mapsto g(z)$ satisfies the following conditions:

- (i) the group action ψ can be extended to a smooth action on $\overline{D}$;
- (ii) for each $z_0 \in \partial D$, the image of the tangent map $(\psi_{z_0})_* : T_eG \to T_{z_0}(\partial D)$ is not a subset of $T^{\mathbb{C}}_{z_0}(\partial D)(:= T_{z_0}(\partial D) \cap iT_{z_0}(\partial D))$.

Furthermore, if the action ψ can be extended to a smooth action on a neighbourhood $V \subset \mathrm{Aut}(D) \times \mathbb{C}^n$ of $\overline{G \times D}$ such that it is holomorphic in the z variable, then we say that D has *transverse symmetries extending beyond the boundary*.

Barrett [Bar82] studied the above notion and proved that any domain that has transverse symmetries automatically satisfies Condition R. In particular, this proves that all bounded Reinhardt domains with smooth boundaries satisfy condition R (note that pseudoconvexity is *not* assumed). Furthermore, Barrett has shown that if a domain has transverse symmetries extending beyond the boundary, then one can also extend proper holomorphic mappings beyond the boundary. More precisely, we have the following:

Theorem 1.2.8 (Barrett [Bar82]). *Let D and D' be bounded domains with smooth boundaries that have transverse symmetries extending beyond the boundary. Then any proper holomorphic mapping $F : D \to D'$ extends to a holomorphic mapping in a neighbourhood of $\overline{D}$.*

All the results surveyed so far exploit the Bergman kernel in their proofs. All of them also assume that the domains under consideration have smooth boundaries. A key tool that is used in many of the proofs is the transformation formula for the Bergman kernel and projection (Result 2.2.2). However, this transformation formula is valid for all bounded domains regardless of boundary smoothness. Thus, the machinery of the Bergman kernel and Bergman projection can be deployed even in the absence of boundary smoothness to extend proper holomorphic mappings. The following result of Bell illustrates this theme.

Theorem 1.2.9 (Bell [Bel82b]). *Let D_1 and D_2 be bounded circular domains $\mathbb{C}^n$ that contain the origin. If $f : D_1 \to D_2$ is a proper holomorphic mapping such that $f^{-1}\{0\} = \{0\}$, then f is a polynomial mapping.*

The methods used in the proof of the above theorem can be used to prove a general meta-result for the extension of proper holomorphic mappings between circular domains; see Result 2.2.3. A consequence of this meta-theorem is used in all of our rigidity results, and it is presented in Section 2.2.

We conclude this section with a brief discussion of the reflection principle. As the reflection principle is not used anywhere in our work, but is only important in the present context to set the stage for our discussion of rigidity results, we shall be very brief and we will not give precise statements of results. The interested reader can refer to Section 2 of [For93] for the early results on the reflection principle, and to [DP09] for the latest developments.

Let D, D' be bounded domains in $\mathbb{C}^n$ whose boundaries are real-analytic, and let $f : D \to D'$ be a proper holomorphic mapping. If $n = 1$, one can extend f holomorphically beyond the boundary of D by applying the classical reflection principle on small portions of the boundary. The analogous question of whether f extends holomorphically beyond the boundary of D if $n > 1$ remains open. The reflection principle in higher dimensions was introduced by Pinchuk in [Pin75]. It has been refined by several authors, and significant progress has been made in the resolution of the above question by using the reflection principle. For instance, the answer to the above question is "Yes," if D and D' are in $\mathbb{C}^2$ (see [DP95]). The answer is also "Yes" in higher dimensions, if D and D' are both assumed to be pseudoconvex (see [DF88]). The latest result of Diederich and Pinchuk [DP03] in fact gives a positive answer, without any additional hypothesis on ∂D, $\partial D'$, assuming only that f extends continuously to ∂D.

This concludes our survey of results on boundary regularity of proper holomorphic mappings. We are now in a position to survey rigidity results of proper holomorphic mappings of domains whose boundaries have some smoothness.

1.3 The Alexander phenomenon

The unit ball in $\mathbb{C}^n$, denoted by $\mathbb{B}^n$, is a very special domain from the perspective of function theory. It has various nice attributes: it has a smooth real-analytic boundary, it is pseudoconvex and of finite type, it is a bounded symmetric domain and hence homogeneous, it is balanced and convex, it is Reinhardt, etc. The precise automorphism group of $\mathbb{B}^n$ has been known since the time of Poincaré. When $n = 1$, one also knows that the finite Blaschke products are the only proper holomorphic self-maps. In contrast, when $n \geq 2$, it was not known for a long time whether there is *even one* non-injective proper holomorphic self-map of $\mathbb{B}^n$! The non-existence of non-trivial proper holomorphic self-mappings of $\mathbb{B}^n, n \geq 2$, was proved by Alexander in 1977.

Theorem 1.3.1 (Alexander [Ale77]). *Any proper holomorphic self-mapping of* $\mathbb{B}^n$, $n > 1$, *is an automorphism.*

Slightly before the work of Alexander, Pinchuk had established the following:

Theorem 1.3.2 (Pinchuk [Pin74]). *Let* D_1, $D_2 \subset \mathbb{C}^n$, $n > 1$, *be bounded strictly pseudoconvex domains, and let* $f : D_1 \to D_2$ *be a proper holomorphic mapping. If* f *extends to a* C^1 *mapping on* $\overline{D}_1$, *then* f *is a local biholomorphism. Furthermore, if* $D_1 = D_2$, *then* f *is an automorphism.*

As a part of the proof, Pinchuk established the following:

Theorem 1.3.3. *Let* $D \subset \mathbb{C}^n$, $n > 1$, *be a bounded domain with* C^2*-smooth boundary, and let* $f : D \to D$ *be a proper holomorphic mapping that is unbranched (i.e., a local biholomorphism). If* f *extends to a* C^1 *mapping on* $\overline{D}$, *then* f *is an automorphism of* D.

The above result is of cardinal importance in obtaining Alexander-type theorems when the boundary of the domain under consideration has some smoothness. It is the last step in the proof of a number of such theorems. The results surveyed in Section 1.2 show that the extension hypothesis in Theorems 1.3.2 and 1.3.3 are automatically satisfied for a large class of domains. In particular, an Alexander-type theorem holds true for all bounded strictly pseudoconvex domains with smooth boundary. The following example, discovered independently by both Pinchuk and Siu, shows that one cannot expect such a result to be true, in general, when $D_1 \neq D_2$.

Example 1.3.4. Let

$$
\begin{aligned}
D_1 &= \left\{(z,w) \in \mathbb{C}^2 : |z|^4 + \frac{1}{|z|^4} + |w|^2 < 3\right\} \\
D_2 &= \left\{(z,w) \in \mathbb{C}^2 : |z|^2 + \frac{1}{|z|^2} + |w|^2 < 3\right\},
\end{aligned}
$$

and let $f : D_1 \to D_2$ be given by $f(z,w) = (z^2, w)$. Note that f is a 2-to-1 covering projection. It is also easy to see that both D_1 and D_2 are strictly pseudoconvex.

The results of Alexander and Pinchuk inspired a lot of results of a similar nature; Section 3 of the survey article [For93] presents some of these. These results have in their hypotheses some combination of the various attributes enjoyed by $\mathbb{B}^n$ listed in the beginning of this section. We will now state a few related results on this theme, concentrating primarily on results that were proved after the aforementioned survey article was published.

The following result by Bedford and Bell proves an Alexander-type theorem for a large class of weakly pseudoconvex domains. Their methodology involves stratifying the boundary by means of a function τ, which we shall discuss in detail in Section 4.2. This function will play a central role in one of our rigidity results (Theorem 4.1.1).

Theorem 1.3.5 (Bedford and Bell [BB82])**.** *If $D \subset \mathbb{C}^n$, $n > 1$, is a bounded weakly pseudoconvex domain with smooth real-analytic boundary, then any proper holomorphic self mapping of D is an automorphism.*

Huang and Pan established an Alexander-type theorem for bounded domains with real-analytic boundary (note that pseudoconvexity is *not* assumed).

Theorem 1.3.6 (Huang and Pan [HP96])**.** *Let $D \subset \mathbb{C}^n$, $n > 1$, be a bounded domain with smooth real-analytic boundary. Then any proper self-map of D that extends smoothly to ∂D must be an automorphism.*

Remark 1.3.7. From the results on the reflection principle surveyed in Section 1.2, it seems likely that the extension hypothesis in the above result is actually redundant.

Berteloot obtained a precise description of the Lie algebra of holomorphic tangent vector fields of strictly pseudoconvex Reinhardt hypersurfaces, and used this description to obtain an Alexander-type theorem for bounded complete Reinhardt domains with C^2-smooth boundary.

Theorem 1.3.8 (Berteloot [Ber98])**.** *Let $D \subset \mathbb{C}^n$, $n > 1$, be a bounded complete Reinhardt domain with C^2-smooth boundary. Then every proper holomorphic self-map of D is an automorphism.*

The following result by Coupet, Pan and Sukhov weakens the hypothesis on the symmetries of the domain in question, but requires additional hypotheses on the boundary and also requires the domain to be in $\mathbb{C}^2$. In the two following results, the authors use the phrase "smoothly bounded" to refer to a domain that is bounded and has C^∞-smooth boundary.

Theorem 1.3.9 (Coupet, Pan and Sukhov [CPS99])**.** *Let $D \subset \mathbb{C}^2$ be a smoothly bounded pseudoconvex complete circular domain of finite type. Then every proper holomorphic self mapping of D is an automorphism.*

Coupet, Pan and Sukhov later extended the above theorem to quasi-circular domains.

Theorem 1.3.10 (Coupet, Pan and Sukhov [CPS01]). *Let $D \subset \mathbb{C}^2$ be a smoothly bounded pseudoconvex quasi-circular domain of finite type. Then every proper holomorphic self-map of D is an automorphism.*

The proofs of Theorems 1.3.8-1.3.10 introduce a new idea in the study of the Alexander phenomenon: that of analyzing the behaviour of the iterates of the self-map in question. This is, in our opinion, an extremely promising idea — we elaborate on this thought in Section 1.5. Another result that exploits this idea is the following:

Theorem 1.3.11 (Berteloot and Patrizio [BP00]). *Let $D \subset \mathbb{C}^n$, $n > 1$, be a bounded complete circular domain with C^2-smooth boundary. Let $f : D \to D$ be a proper holomorphic map, and let f_p be the lowest-degree non-constant homogeneous polynomial mapping in the Taylor expansion of f about 0. If $f^{-1}\{0\} = f_p^{-1}\{0\} = \{0\}$, then f is a linear automorphism of D.*

The above results prompt the following natural question (and is also one of the motivations for the widely-discussed Conjecture 1.3.12):

> *What role, if any, do the various attributes of $\mathbb{B}^n$ listed in the beginning of this section play in the phenomenon exhibited in Alexander's theorem?*

Conjecture 1.3.12. *Let $D \subset \mathbb{C}^n$, $n > 1$, be a bounded domain with C^2-smooth boundary. Then every proper holomorphic self-mapping of D is an automorphism.*

Note that the above conjecture is far from being fully settled even with the additional hypothesis of pseudoconvexity. But, as we have seen, a number of results are known if one makes additional assumptions on the boundary and the automorphism group. In Section 1.5, we give an outline of our generalization of Theorem 1.3.9 to higher dimensions.

1.4 Bounded symmetric domains

All the rigidity results that we have discussed so far, impose restrictions on the boundary. One can also ask whether hypotheses *solely* on the automorphism group will also deliver structure results. The following result by Bell gives a positive answer for circular domains that contain the origin.

Theorem 1.4.1 (Bell [Bel93]). *Let D_1 and D_2 be bounded circular domains in $\mathbb{C}^n$ that contain the origin. If $f : D_1 \to D_2$ is a proper holomorphic mapping that fixes the origin, then f is an algebraic mapping. Furthermore, if $f^{-1}\{0\} = \{0\}$, then f is a polynomial mapping. If it is still further assumed that f is a biholomorphism, then f must be linear.*

Remark 1.4.2. Note that the last conclusion of the above theorem is the classical uniqueness theorem of Cartan. The above result can, therefore, be viewed as a very broad generalization of Cartan's uniqueness theorem.

Remark 1.4.3. A version of this result was presented in Section 1.2 (Theorem 1.2.9). The above result is a later work of Bell, and the proof is unique in the sense that it does *not* use the machinery of the Bergman kernel. However, a careful reading of [Bel93] will reveal that a lot of the constructions and many of the lemmas are motivated by Bell's earlier work [Bel82b] that *does* use the Bergman kernel machinery.

Bell's result is remarkable as we are able to obtain quite a strong structure result assuming just circular symmetry. It is quite natural to ask whether a stronger conclusion can be obtained in the presence of more symmetry. The following result due to Henkin and Novikov gives an affirmative answer for a certain class of bounded symmetric domains.

Theorem 1.4.4 (Henkin and Novikov [HN84])**.** *Let $D \subset \mathbb{C}^n$, $n \geq 3$, be an irreducible bounded symmetric domain of Type IV in Cartan's classification. Then every proper holomorphic self-map of D is an automorphism.*

Before we continue our survey, we pause and give a definition.

Definition 1.4.5. Let $D \subset \mathbb{C}^n$ be a bounded domain. We say that D is symmetric if for each $a \in D$, we can find an involutive automorphism s_a of D such that a is an isolated fixed point of D. We say that D is *irreducible* if it is not biholomorphic to a product of lower-dimensional domains.

Remark 1.4.6. The definition of a bounded symmetric domain we have given is rather *ad hoc*. In the language of differential geometry, a bounded symmetric domain is a realization in $\mathbb{C}^n$, as a bounded domain, of a Hermitian symmetric space of non-compact type.

Cartan has given a complete classification of irreducible bounded symmetric domains, showing that they are one of six types of homogeneous spaces. An outcome of Harish-Chandra's work is that these homogeneous spaces (and products thereof) can be imbedded in $\mathbb{C}^n$ as bounded convex balanced domains (hence, they are realizable as open unit balls relative to some $\mathbb{C}$-norm). This imbedding is unique up to a linear isomorphism of $\mathbb{C}^n$. Such a realization of a bounded symmetric domain is called the *Harish-Chandra realization*.

The result of Henkin and Novikov (Theorem 1.4.4) proves that an Alexander-type theorem holds true for domains of Type IV in Cartan's classification. One would like to know whether this is true for all six types. Henkin and Novikov make a remark at the end of [HN84] that Bell's result (Theorem 1.4.1) in combination with [TK82] in fact establishes Theorem 1.4.4 for *all* irreducible bounded symmetric domains other than $\mathbb{D}$. We could not find any work in the literature that works out the details of this remark. However, one can show that, with the use of the aforementioned results — and following the scheme of the proof of Theorem 1.4.4 — the conclusion of Theorem 1.4.4 *does* extend to a larger subclass of the bounded symmetric domains.

About a decade after the work of Henkin and Novikov, Tsai established the following theorem, which settled a conjecture of Mok [Mok89, Chapter 6, (5.3)].

Theorem 1.4.7 (Tsai [Tsa93])**.** *Let D_1 and D_2 be two irreducible bounded symmetric domains and let $f : D_1 \to D_2$ be a proper holomorphic mapping. Furthermore, let* $\operatorname{rank}(D_1) \geq \operatorname{rank}(D_2) \geq 2$*. Then f is a totally geodesic isometric embedding (up to a normalization constant) with respect to the Bergman metrics on D_1 and D_2.*

Adapting Tsai's ideas to the equidimensional case, Tu established the following result.

Theorem 1.4.8 (Tu [Tu02])**.** *Let D_1 and D_2 be two equidimensional bounded symmetric domains. Assume that D_1 is irreducible and* $\operatorname{rank}(D_1) \geq 2$*. Then, any proper holomorphic mapping $f : D_1 \to D_2$ is a biholomorphism.*

In using Tsai's ideas, the assumption that D_1 is irreducible, is essential — see [Tu02, Proposition 3.3] — and it is not clear that a small mutation of those ideas allows one to weaken this assumption. On the other hand, the methods in [HN84] (and [TK82], on which [HN84] relies), [Tsa93] and [Tu02] are tied, in a rather maximalistic way, to the fine structure of a bounded symmetric domain. Furthermore, these methods are *not* applicable when D_1 is of rank 1 (i.e., D_1 is biholomorphic to the unit ball).

It is also somewhat unsatisfactory that the methods used to deal with the higher rank case are not complex-analytic in spirit — and rely upon significant machinery from representation theory and metric geometry — whereas one can give a purely complex-analytic proof of Alexander's theorem for the ball. In our work, we present a proof of a rigidity result for bounded symmetric domains that does *not* involve any assumption on rank, and in which it suffices that *either* D_1 or D_2 be irreducible in Theorem 1.4.8. This gives a result that subsumes all the known rigidity results for proper holomorphic maps between equidimensional bounded symmetric domains. Moreover, apart from some classical results on Hermitian symmetric spaces, our proof uses methods primarily from complex analysis. We give a precise statement and brief discussion of our result in the next section.

1.5 A discussion of our results

In this section, we present the statements of our original results, and we highlight some of the novelties in our proofs. Detailed proofs are given in subsequent chapters, and for the convenience of the reader, the numbering of the following results will indicate their positions in the later chapters.

We begin with the statement of our first structure result, which is a descriptive result.

Theorem 3.1.1. *Let R_j and S_j, $j = 1, \dots, n$, be compact Riemann surfaces, and let X_j (resp. Y_j) be a connected, hyperbolic open subset of R_j (resp. S_j) for each $j = 1, \dots, n$. Let $F = (F_1, \dots, F_n) : X_1 \times \cdots \times X_n \to Y_1 \times \cdots \times Y_n$ be a finite proper holomorphic map. Then, denoting $z \in X_1 \times \cdots \times X_n$ as $(z_1, \dots, z_n)$, each F_i is of the form $F_i(z_{\pi(i)})$, where π is a permutation of* $\{1, \dots, n\}$.

Remark 1.5.1. It is essential for F to be a finite map in the above theorem. Without this requirement, Theorem 3.1.1 is false. To see this, let X be some compact hyperbolic Riemann surface. The map $F : X^2 \to X^2$ defined by $F(z_1, z_2) := (z_1, z_1)$ is a proper map. In fact, F satisfies all the assumptions of Theorem 3.1.1 except finiteness.

Remark 1.5.2. The conclusion of the above theorem can fail if even one of the factors is non-hyperbolic. Consider $X = \mathbb{D} \times (\widehat{\mathbb{C}} \setminus \{p\})$, where $p \in \widehat{\mathbb{C}}$ and $\mathbb{D}$ denotes the unit disc in $\mathbb{C}$. We know that $\widehat{\mathbb{C}} \setminus \{p\}$ is **not** hyperbolic. We view $\widehat{\mathbb{C}} \setminus \{p\}$ as $\mathbb{C}$. It is easy to check that any $F \in \mathrm{Aut}(\mathbb{D} \times \mathbb{C})$ is of the form

$$F(z_1, z_2) = (\psi(z_1), A(z_1)z_2 + B(z_1)),$$

where $\psi \in \mathrm{Aut}(\mathbb{D}), A, B \in \mathcal{O}(\mathbb{D})$ and A is non-vanishing.

Our motivation for considering connected hyperbolic open subsets of compact Riemann surfaces comes from the illustrative example, Example3.1.2, presented in Chapter 3. The novelty of our proof, from the viewpoint of function theory, lies in our use of the fact that the set of non-constant holomorphic maps between certain Riemann surfaces is at most finite. This phenomenon is well understood in the realm of *compact* complex manifolds; see, for instance, [Kob98, Chapters 6 & 7]. However, the factors X_j and Y_j in Theorem 3.1.1 are not necessarily compact. We will see that the main idea in the Remmert–Stein theorem (i.e. Result 1.1.2) is still useful in our more general setting. Loosely speaking, we show that, in general, the manifold $X_1 \times \cdots \times X_n$ splits into two factors, one of which is the product of those non-compact factors to which the Remmert–Stein method can be applied. The finiteness result that is essential to our proof is a result by Imayoshi [Ima83]. We give a detailed proof of Theorem 3.1.1 in Chapter 3.

We now come to rigidity results. Before we state our first result, we make a preliminary definition.

Definition 1.5.3. We say that a domain $D \subset \mathbb{C}^n$ is *circular* if, for any $z \in D$, $e^{i\theta}z \in D$, $\forall \theta \in [0, 2\pi)$. We say that it is *balanced* if, for any $z \in D$, $\zeta z \in D$ for each $\zeta \in \overline{\mathbb{D}}$.

Remark 1.5.4. Balanced domains are also called *complete circular domains* in the literature. In stating the results of this thesis, we shall use the term "balanced" rather than the term "complete circular" as we feel that the adjective *complete* is used to denote too many things in the literature.

Our first rigidity result is the following theorem. We point out that this theorem extends the result of Coupet–Pan–Sukhov (Theorem 1.3.9 above) to higher dimensions.

Theorem 4.1.1. *Let $\Omega \subset \mathbb{C}^n$, $n > 1$, be a smoothly bounded pseudoconvex balanced domain of (D'Angelo) finite type. Then every proper holomorphic self mapping $F : \Omega \to \Omega$ is an automorphism.*

The proofs of both Theorem 1.3.9 and Theorem 4.1.1 share some key ideas. Central to both is the need for a proposition that gives a precise description of the branch locus of a proper holomorphic self-mapping of the domain under consideration in Theorem 4.1.1, under the *assumption* that this map is branched. In our setting, a proposition that suffices is as follows:

Proposition 4.1.2. *Let $\Omega \subset \mathbb{C}^n$, $n > 1$, be a smoothly bounded pseudoconvex balanced domain of (D'Angelo) finite type. Let $F : \Omega \to \Omega$ be a proper holomorphic mapping, and assume that the branch locus $V_F := \{z \in \Omega : \mathrm{Jac}_{\mathbb{C}} F(z) = 0\} \neq \emptyset$. Let X be an irreducible component of V_F. Then for each $z \in X$, the set $(\mathbb{C} \cdot z) \cap \Omega$ is contained in X.*

Needless to say, the strategy of our proof of Theorem 4.1.1 will be to use the above proposition to reach a contradiction.

Coupet, Pan and Sukhov have proved a version of Proposition 4.1.2 for domains in $\mathbb{C}^2$ in which the domain need not be balanced, but is only required to admit a transverse $\mathbb{T}$-action. The point that is worth highlighting here is that by restricting ourselves to balanced domains, we are able to give an almost entirely elementary proof, and that these methods have one significant payoff: we *do not* have to assume in the above results that the $\mathbb{T}$-action is transverse. We bypass the need for transversality by using some results from dimension theory. These results are insensitive to dimension, which allows us to state and prove Theorem 4.1.1 in $\mathbb{C}^n$ for all $n > 1$.

A key tool used in the proofs of Proposition 4.1.2 and Theorem 4.1.1 is the function $\tau : \partial\Omega \to \mathbb{Z}_+ \cup \{0\}$ introduced by Bedford and Bell (see [BB82, Bel84a]). The number $\tau(p)$ is the order of vanishing in the tangential directions of the Levi determinant of a smoothly bounded pseudoconvex domain at the point $p \in \partial\Omega$. The function τ has been successfully used to study the branching behaviour of proper holomorphic mappings in many earlier results.

Using Proposition 4.1.2, one can prove that $F^{-1}\{0\} = \{0\}$. It is at this point that our proof and the proof given by Coupet–Pan–Sukhov diverge. For domains in $\mathbb{C}^2$, with the notation as in Proposition 4.1.2, it is easy to see that V_F must be a finite union of disks. It follows from this, in $\mathbb{C}^2$, that F is a homogeneous polynomial map. Using the last three facts and a result by Hubbard and Papadopol [HP94], Coupet–Pan–Sukhov conclude the $\partial\Omega$ must be non-smooth. From this contradiction, they infer that V_F cannot be non-empty. We remark in passing that, having proved that F is a homogeneous polynomial mapping, the fact that F must be an automorphism follows at once from Theorem 1.3.11.

Our approach to finishing the proof of Theorem 4.1.1 also relies on tools from complex dynamics. Motivated by the fact that a number of rigidity results on circular domains use tools from complex dynamics, we searched the literature for results or techniques in complex dynamics that would be effective in higher dimensions. A recent reuslt of Opshtein gives us the tools that we need. Refer to Section 4.3 for the definitions of the terms that occur in the following result:

Result 1.5.5 (Opshtein [Ops06], Théorème A and Remarque 30)**.** *Let $D \subset \mathbb{C}^n$, $n > 1$, be a smoothly bounded pseudoconvex domain whose boundary is B-regular. Let $f : D \to D$ be a proper holomorphic self-map that is recurrent. Then the limit manifold of f is necessarily of dimension higher than* 1.

In [Ops06], Opshtein suggests that his results could serve as a new set of tools for establishing Alexander-type theorems. He has also established a slight generalization of Theorem 1.3.9 using these tools ([Ops06, Théorème C]). We found Opshtein's viewpoint very useful in the context of Theorem 4.1.1.

Finally, we present the statements of our rigidity results on bounded symmetric domains.

Theorem 5.1.1. *Let D_1 and D_2 be two bounded symmetric domains of complex dimension $n \geq 2$. Assume that either D_1 or D_2 is irreducible. Then, any proper holomorphic mapping of D_1 into D_2 is a biholomorphism.*

As we had mentioned in Section 1.4, the above result subsumes all the other known rigidity results on equidimensional bounded symmetric domains. We now highlight a couple of features of our work, contrasting it with some of the results surveyed in Section 1.4.

a) The techniques underlying [Tsa93] and [Tu02] rely almost entirely on the fine structure of a bounded symmetric domain. Specifically, they involve studying the effect of a proper holomorphic map on the characteristic symmetric subspaces of a bounded symmetric domain of rank ≥ 2. In contrast, our techniques rely on only a coarse distinction between the different strata that comprise the boundary of an irreducible bounded symmetric domain.

b) An advantage of arguments that rely on only a coarse resolution of a bounded symmetric domain is that some of them are potentially applicable to the study of domains that have non-compact automorphism groups, but are not assumed to be symmetric. A demonstration this viewpoint is the proof of Theorem 5.1.3 below.

Let D be a bounded symmetric domain. The main technical tool that facilitates our study of the structure of ∂D, and describes certain elements of $\mathrm{Aut}(D)$ with the optimal degree of explicitness, is the notion of Jordan triple systems. The application of Jordan triple systems to geometry appears to have been pioneered by Koecher [Koe99]. Our reference on this subject are the lecture notes of Loos [Loo77], which are devoted specifically to the bounded symmetric domains. Jordan triple systems and versions of the Schwarz lemma are our primary tools. Given a bounded symmetric in domain in its Harish-Chandra realization, one can, using the Bergman kernel and metric, associate to it a triple product on $\mathbb{C}^n$ that satisfies certain special properties. Geometric properties of the bounded symmetric domain get translated into algebraic properties of the associated triple system. For instance, one can obtain a precise description of the boundary geometry of a bounded symmetric domain. One also obtains descriptions of the Shilov boundary of a bounded symmetric domain. These descriptions are obtained in a manner that does not discriminate between the unit (Euclidean) ball in $\mathbb{C}^n$, $n \geq 2$, and the higher-rank domains. This is one feature that distinguishes our proof from those in [Tsa93] and [Tu02].

The machinery of Jordan triple systems also furnishes formulas for certain special automorphisms of bounded symmetric domains (the analogues of the classical Möbius transformations on $\mathbb{D}$ that map the origin to points in $\mathbb{D}\setminus\{0\}$). While the *specific* formulas can be very complicated, their *general* descriptions in terms of operators described by the triple-system machinery are such that one can make a universal estimate — irrespective of the specific domain or its rank — on the extensions of these automorphisms to the boundary. The ability to extend the aforementioned automorphisms, with *good estimates*, to the boundary is key to our proof — we elaborate a bit more on this thought in Section 5.1.

We now state a result that was obtained in collaboration with Gautam Bharali. The result demonstrates the application of some of the techniques used in the proof of Theorem 5.1.1 in more general contexts. We first need a few definitions.

Definition 1.5.6. Let $D \subsetneq \mathbb{C}^n$ be a domain and let $p \in \partial D$. We say that p is a *peak point* if there exists a function $h \in \mathcal{O}(D) \cap C(\overline{D}; \mathbb{C})$ such that $h(p) = 1$ and $|h(z)| < 1\ \forall z \in \overline{D} \setminus \{p\}$. The function h is called a *peak function for p*.

Definition 1.5.7. Let $D \subsetneq \mathbb{C}^n$ be a domain and let $p \in \partial D$. We say that p is a *boundary orbit-accumulation point* if there exist a point $a \in D$ and a sequence of automorphisms $\{\phi_k\}$ of D such that $\lim_{k\to\infty} \phi_k(a) = p$.

Remark 1.5.8. When a domain D is bounded, the non-compactness of $\mathrm{Aut}(D)$ (in the compact-open topology) is equivalent to D having a boundary orbit-accumulation point; see [Nar71].

With the last two definitions, we are in a position to state our second theorem on bounded symmetric domains. Note that D_1 is *not* assumed to be a bounded symmetric domain. Yet, some of the techniques (versions of which have been used to remarkable effect in the literature in this field) used in the proof of Theorem 5.1.1 are general enough to be applicable to the following situation.

Theorem 5.1.3. *Let D_1 be a bounded convex balanced domain in $\mathbb{C}^n$ whose automorphism group is non-compact and let p be a boundary orbit-accumulation point. Let D_2 be a realization of a bounded symmetric domain as a bounded convex balanced domain in $\mathbb{C}^n$. Assume that there is a neighbourhood U of p and a biholomorphic map $F : U \to \mathbb{C}^n$ such that $F(U \cap D_1) \subset D_2$ and $F(U \cap \partial D_1) \subset \partial D_2$. Assume that either p or $F(p)$ is a peak point. Then, there exists a* linear *map that maps D_1 biholomorphically onto D_2.*

Remark 1.5.9. Theorem 5.1.3 (together with Bell's theorem [Bel82b]) gives a very short proof of the rigidity theorem of Mok and Tsai [MT92] under the additional assumption that the convex domain D in their result is also circular. There is an extensive literature on rigidity theorems relating to bounded symmetric domains, but we shall not dwell any further on it.

2 Background Results

The purpose of this chapter is to present, in one convenient place, a number of definitions and results that we will use in more than one occasion in what follows. We will not prove all the results, but we will give precise references. We begin by presenting the basic definitions and results pertaining to proper holomorphic mappings in Section 2.1. Section 2.2 summarizes the basic properties of the Bergman kernel. We also present a result of Bell on proper holomorphic mappings of circular domains in this section. Finally, in Section 2.3, we present material on complex geodesics, which have proven to be a very powerful tool in the study of holomorphic mappings.

2.1 Proper holomorphic mappings

The standard reference for this section is [Rud08, Chapter 15]. We begin with the definition of a proper holomorphic mapping.

Definition 2.1.1. Let X and Y be topological spaces. A continuous map $F : X \to Y$ is said to be *proper* if $F^{-1}(K)$ is compact in X for every compact $K \subseteq Y$.

In the case where $F : D \to D'$ is a proper map, and D and D' are domains in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, this is equivalent to the requirement that for every sequence $\{z_i\}$ in D that has no limit point in D, $\{F(z_i)\}$ has no limit point in D'. By the definition of properness, it also follows that the inverse image of a point $w \in D'$ under F is a compact complex-analytic subvariety of D, and hence must be finite. Thus, proper holomorphic mappings between domains are finite mappings. This, combined with the rank theorem, also shows that if $m < n$, then there cannot be a proper holomorphic mapping from D into D'.

The next result summarizes the basic properties of a proper holomorphic mappings between domains of the same dimension.

Result 2.1.2. *Let D, $D' \subset \mathbb{C}^n$ be domains, and let $F : D \to D'$ be a proper holomorphic mapping. Then:*

(i) *$F(D) = D'$;*

(ii) *the regular values of F form a connected open set that is dense in D';*

(iii) *there is an integer m such that $\#(F^{-1}\{w\}) = m$ when $w \in D'$ is a regular value and is independent of w, and $\#(F^{-1}\{w\}) < m$ when $w \in D'$ is a critical value;*

(*iv*) *$F(V)$ is a complex-analytic subvariety of D' whenever V is a complex-analytic subvariety of D.*

2.2 The Bergman kernel and Bell's theorem

Let $D \subset \mathbb{C}^n$ be a bounded domain. We are interested in the space $B(D) := L^2(D) \cap \mathcal{O}(D)$. It is a standard fact that $B(D)$ is a closed subspace of $L^2(D)$, and is therefore a Hilbert space in its own right. It is also easy to see that for a fixed $a \in D$, the evaluation map

$$\tau_a : B(D) \ni f \mapsto f(a) \in \mathbb{C},$$

is a continuous linear functional. Therefore, by the Riesz representation theorem, there is a unique element in $B(D)$, say $K_D(\cdot, a)$, such that

$$\tau_a(f) = \int_D f(w)\overline{K_D(w,a)}\, dV(w).$$

The function $K_D : D \times D \to \mathbb{C}$ thus defined is called the *Bergman kernel* of D. The following result summarizes the basic properties of the Bergman kernel.

Result 2.2.1. *Let $D \subset \mathbb{C}^n$ be a bounded domain, and let K_D be its associated Bergman kernel. Then,*

(*i*) *$K_D(w,z)$ is holomorphic in w and conjugate holomorphic in z. In fact, we have*

$$K_D(w,z) = \overline{K_D(z,w)}.$$

(*ii*) *For any orthonormal basis $\{\phi_j : j = 1,2,\ldots\}$ of $B(D)$, we have the representation*

$$K_D(w,z) = \sum_{j=1}^{\infty} \phi_j(w)\overline{\phi_j(z)} \;\; \forall (w,z) \in D \times D,$$

where the convergence is uniform on compact subsets of $D \times D$.

(*iii*) *Let $P_D : L^2(D) \to B(D)$ be the orthogonal projection. Then, P_D satisfies*

$$(P_D f)(w) = \int_D f(z) K_D(w,z)\, dV(z) \;\; \forall f \in L^2(D),\ \forall w \in D.$$

We now state a result of Bell that describes the transformation of the Bergman kernel and the Bergman projection under a proper holomorphic mapping. This transformation formula is one of the most important tools in the study of proper holomorphic mappings.

Result 2.2.2 (Bell [Bel82a]). *Let D_1 and D_2 be bounded domains with associated Bergman kernels K_1 and K_2 respectively. Let $f : D_1 \to D_2$ be a proper holomorphic mapping. Let V be the set of critical values of f. Then,*

$$\sum_{z \in f^{-1}\{\zeta\}} \frac{K_1(w,z)}{\overline{\mathsf{Jac}_{\mathbb{C}}(f)(z)}} = \mathsf{Jac}_{\mathbb{C}}(f)(w) K_2(f(w),\zeta) \;\; \forall w \in D_1, \forall \zeta \in D_2 \setminus V. \tag{2.1}$$

The transformation formula for the Bergman projection is given by

$$\mathsf{Jac}_{\mathbb{C}}(f) \cdot (P_2 g) \circ f = P_1 \left(\mathsf{Jac}_{\mathbb{C}}(f) \cdot g \circ f\right) \;\; \forall g \in L^2(D_2). \tag{2.2}$$

We shall now present an application of the transformation formula to proper holomorphic mappings of circular domains.

Result 2.2.3 (Bell [Bel82b]). *Suppose $f : D_1 \to D_2$ is a proper holomorphic map between bounded circular domains. Suppose further that D_2 contains the origin and that the Bergman kernel $K(w,z)$ associated to D_1 is such that for each compact subset G of D_1, there is an open set $U = U(G)$ containing $\overline{D}_1$ such that $K(\cdot, z)$ extends to be holomorphic on U for each $z \in G$. Then f extends holomorphically to a neighbourhood of $\overline{D}_1$.*

The above result is presented at the end of the paper cited above, without proof. It is clear that no proof is given because its proof follows *mutatis mutandis* from the arguments presented in [Bel82b] and [Bel81]. For the reader's convenience, we present a proof here. We begin with a lemma from [Bel82b].

Lemma 2.2.4 (Lemma C of [Bel82b]). *Suppose $D \subset \mathbb{C}^n$ is a bounded circular domain that contains the unit ball $\mathbb{B}^n$. Let P be the Bergman projection associated to D. For each multi-index $\alpha \in \mathbb{N}^n$, there is a function $\phi_\alpha \in C_0^\infty(\mathbb{B}^n)$ such that $P\phi_\alpha = z^\alpha$.*

The proof of Result 2.2.3. Without loss of generality, we may assume that the unit ball is contained in D_2. Fix a multi-index $\alpha \in \mathbb{N}^n$, and let ϕ_α be the function from Lemma 2.2.4 associated to D_2. From the transformation rule for the Bergman projection (2.2), we get

$$\mathsf{Jac}_{\mathbb{C}}(f) \cdot f^\alpha = \mathsf{Jac}_{\mathbb{C}}(f) \cdot (z^\alpha \circ f) = P_1 \left(\mathsf{Jac}_{\mathbb{C}}(f) \cdot (\phi_\alpha \circ f)\right),$$

where P_1 denotes the Bergman projection associated to D_1. In the integral form, we have :

$$\mathsf{Jac}_{\mathbb{C}}(f)(w) \cdot f^\alpha(w) = \int_{D_1} K_1(w,z) \mathsf{Jac}_{\mathbb{C}}(f)(z) \phi_\alpha(f(z)) \, dV(z). \tag{2.3}$$

Note that, for a fixed w,

$$\mathrm{Supp}\left(K_1(w,z)\mathsf{Jac}_{\mathbb{C}}(f)(z)\phi_\alpha(f(z))\right) \subset f^{-1}(\mathrm{Supp}(\phi_\alpha)).$$

As $\mathrm{Supp}(\phi_\alpha)$ is a compact set, it follows from properness that $G_\alpha := f^{-1}(\mathrm{Supp}(\phi_\alpha))$ is also a compact set. By hypothesis, there exists an open set $U_\alpha \supset \overline{D}_1$ such that the $K_1(\cdot, z)$ extends to be holomorphic on U_α for each $z \in G_\alpha$. Therefore, it follows from (2.3) that for each multi-index α, $\mathsf{Jac}_{\mathbb{C}}(f)(w) \cdot f^\alpha(w)$ extends to be holomorphic on U_α. This proves that each multi-index α, $\mathsf{Jac}_{\mathbb{C}}(f) \cdot f^\alpha \in \mathcal{O}(\overline{D}_1)$, the ring of holomophic functions that extend to some neighbourhood of $\overline{D}_1$. Note that this ring is a UFD. We will now show that $f_i \in \mathcal{O}(\overline{D}_1)$, where f_i is some component of f. We have $\mathsf{Jac}_{\mathbb{C}}(f) \cdot f_i^k \in \mathcal{O}(\overline{D}_1)$ $\forall k \in \mathbb{Z}_+ \cup \{0\}$. Consequently, $\mathsf{Jac}_{\mathbb{C}}(f)^{k-1}$ divides $(\mathsf{Jac}_{\mathbb{C}}(f) \cdot f_i)^k$ in $\mathcal{O}(\overline{D}_1)$ for any $k \in \mathbb{Z}_+$. Let $g \in \mathcal{O}(\overline{D})$ be an irreducible factor of $\mathsf{Jac}_{\mathbb{C}}(f)$, and suppose that g^s divides $\mathsf{Jac}_{\mathbb{C}}(f)$. Then $g^{s(k-1)}$ divides $\mathsf{Jac}_{\mathbb{C}}(f)^{k-1}$, and hence g must appear at least $\lceil s(k-1)/k \rceil$ times in the unique factorization of $\mathsf{Jac}_{\mathbb{C}}(f) \cdot f_i$, where$\lceil s(k-1)/k \rceil$ is the smallest integer greater than or equal to $s(k-1)/k$. Since k is an arbitrary positive integer, it follows that g^s divides $\mathsf{Jac}_{\mathbb{C}}(f) \cdot f_i$ in $\mathcal{O}(\overline{D}_1)$. Repeating the same argument for the other irreducible factors, we conclude that $\mathsf{Jac}_{\mathbb{C}}(f)$ divides $\mathsf{Jac}_{\mathbb{C}}(f) \cdot f_i$ in $\mathcal{O}(\overline{D}_1)$, proving that $f_i \in \mathcal{O}(\overline{D}_1)$. This proves that f extends holomorphically to some neigbhourhood of $\overline{D}_1$, and we are done. □

Now let D be any bounded balanced domain (not necessarily convex) in $\mathbb{C}^n$. If D is not convex, it will not be a unit ball with respect to some norm on $\mathbb{C}^n$. However, we do have a function that has the same homogeneity property as a norm, with respect to which D is the "unit ball". The function $M_D : \mathbb{C}^n \to [0, \infty)$ defined by

$$M_D(z) := \inf\{t > 0 : z/t \in D\}$$

is called the *Minkowski functional for D*. Assume that the intersection of each complex line passing through $0 \in \mathbb{C}^n$ with ∂D is a circle. Let G be a compact subset of D. Then, as M_D is upper semicontinuous, $\exists r_G \in (0,1)$ such that $G \subset \{z \in \mathbb{C}^n : M_D(z) < r_G\}$ and the latter is an open set. Hence $z/r_G \in D$ $\forall z \in G$. Clearly, $r_G w \in D$ $\forall w \in \{z \in \mathbb{C}^n : M_D(z) < 1/r_G\} =: U(G)$. By our assumptions, $\overline{D} \subset U(G)$. Let K_D be the Bergman kernel of D. We recall that:

$$K_D(w, z) = \sum_{j=1}^{\infty} \psi_j(w)\overline{\psi_j(z)} \quad \forall (w, z) \in D \times D,$$

where the right-hand side converges absolutely and uniformly on any compact subset of $D \times D$ and $\{\psi_j : j = 1, 2, 3, \dots\}$ is *any* complete orthonormal system for $B(D)$. Then — owing to the fact that the collection $\{C_\alpha z^\alpha : \alpha \in \mathbb{N}^n\}$ (where $C_\alpha > 0$ are suitable normalization constants) is a complete orthonormal system for $B(D)$ — we can infer two things. First: the functions

$$\phi_z(w) := K_D(r_G w, z/r_G), \;\; w \in U(G), \tag{2.4}$$

are well-defined by power series for each $z \in G$. Secondly:

$$K_D(r_G w, z/r_G) = K_D(w, z) \;\; \forall (w, z) \in D \times G.$$

Comparing this with (2.4), we see that each ϕ_z extends $K_D(\cdot, z)$ holomorphically. In view of Result 2.2.3, we have just deduced:

Lemma 2.2.5. *Let $f : D_1 \to D_2$ be a proper holomorphic map between bounded balanced domains. Assume that the intersection of every complex line passing through 0 with ∂D_1 is a circle. Then f extends holomorphically to a neighbourhood of $\overline{D}_1$.*

The above lemma is used in the proofs of all our rigidity results.

2.3 Complex geodesics

In what follows, given a domain $D \subset \mathbb{C}^n$, we will denote the Kobayashi pseudo-distance by K_D, the Carathéodory pseudo-distance by C_D and the infinitesimal versions by κ_D and γ_D respectively. We will denote the Poincaré metric and distance on $\mathbb{D}$ by ω and $\mathsf{p}_{\mathbb{D}}$, respectively. For a detailed treatment of invariant metrics refer to [JP93]. It follows from the definitions that $C_D(x, y) \leq K_D(x, y)$ and $\gamma_D(x; v) \leq \kappa_D(x; v)$ $\forall x, y \in D$ and $v \in \mathbb{C}^n$.

The notion of complex geodesics was introduced by Vesentini ([Ves81, Ves82]) and is a useful tool in the study of holomorphic mappings.

Definition 2.3.1. Let $D \subset \mathbb{C}^n$ be a domain, and let $\phi : \mathbb{D} \to D$ be a holomorphic map.

(*i*) Let $a, b \in D$. The map ϕ is said to be a *d_D-geodesic for* (a, b) if there exist points $x, y \in \mathbb{D}$ such that $\phi(x) = a$, $\phi(y) = b$, and $\mathsf{p}_{\mathbb{D}}(x, y) = d_D(a, b)$, where d_D is either K_D or C_D.

(*ii*) Let $a \in D$ and $V \in \mathbb{C}^n$. The map ϕ is said to be a *δ_D-geodesic for* (a, V) if there exists a number $\alpha \in \mathbb{C}$ and a point $x \in \mathbb{D}$ such that $\phi(x) = a$, $V = \alpha\phi'(x)$, and $\delta_D(a; V) = \omega(x; \alpha)$, where δ_D is either γ_D or κ_D.

We need one more notion before we can state the main result of this section.

Definition 2.3.2. Let $D \subset \mathbb{C}^n$ be a bounded domain. We say that a point $a \in \partial D$ is *holomorphically extreme* if there is no non-constant holomorphic mapping $\phi : \mathbb{D} \to \overline{D}$ such that $\phi(0) = a$. We say that a is a *complex extreme* point if the only vector $b \in \mathbb{C}^n$ for which $a + (b \cdot \overline{\mathbb{D}}) \subset \overline{D}$ is $b = 0$.

Remark 2.3.3. Note that every holomorphically extreme point is automatically complex extreme. It is known that the two notions are equivalent for convex domains.

The following uniqueness result for K_D-geodesics illustrates the importance of the above definition; see [JP93, Proposition 8.3.5] for a proof.

Result 2.3.4. *Let $D \subset \mathbb{C}^n$ be a bounded balanced pseudoconvex domain, and let $a \in D, a \neq 0$, be such that $a/M_D(a) \in \partial D$ is holomorphically extreme, where M_D is the Minkowski functional of D. Then the mapping*

$$\phi_a : \mathbb{D} \ni \lambda \mapsto \lambda a/M_D(a)$$

is the unique (modulo $\mathrm{Aut}(\mathbb{D})$*) K_D-geodesic (κ_D-geodesic) for* $(0, a)$ *(resp.,* $(0, a/M_D(a))$*).*

3 Proper holomorphic mappings of hyperbolic product manifolds

In this chapter, we prove a result on the structure of finite proper holomorphic mappings between complex manifolds that are products of hyperbolic Riemann surfaces. While an important special case of our result follows from the ideas developed by Remmert and Stein, the proof of the full result relies on the interplay of the latter ideas and a finiteness theorem for Riemann surfaces.

Most of the material presented below also appears in the preprint [Jan11].

3.1 Introduction and a motivating example

The main theorem of this chapter is:

Theorem 3.1.1. *Let R_j and S_j, $j = 1, \ldots, n$, be compact Riemann surfaces, and let X_j (resp. Y_j) be a connected, hyperbolic open subset of R_j (resp. S_j) for each $j = 1, \ldots, n$. Let $F = (F_1, \ldots, F_n) : X_1 \times \cdots \times X_n \to Y_1 \times \cdots \times Y_n$ be a finite proper holomorphic map. Then, denoting $z \in X_1 \times \cdots \times X_n$ as $(z_1, \ldots, z_n)$, each F_i is of the form $F_i(z_{\pi(i)})$, where π is a permutation of $\{1, \ldots, n\}$.*

We draw the reader's attention to Remarks 1.5.1 and 1.5.2 for a discussion on why the assumptions of finiteness and hyperbolicity above are essential.

The above result generalizes the classical Remmert–Stein theorem (Theorem 1.1.2). As we had mentioned in Section 1.5, we use a finiteness theorem of Imayoshi to deal with those factors that are compact or compact but for finitely many punctures. The following example motivates, through a special case, the need for a finiteness theorem in circumstances where the techniques used by Remmert and Stein are not applicable.

Example 3.1.2. *Let $D = \mathbb{C} \setminus \{0, 1\}$, and $f = (f_1, f_2)$ be a proper holomorphic self-map of $D \times D$. Note that D is hyperbolic. Even though most of the hypotheses of the Remmert–Stein theorem are not satisfied, its conclusion still follows.*

This is not hard to see. Fix $z_0 \in \mathbb{C} \setminus \{0, 1\}$. By the big Picard theorem, it follows that $0, 1$ and ∞ are removable singularities or poles of the map $h := f_1(z_0, \cdot)$. Hence h extends as a holomorphic map to $\widehat{\mathbb{C}}$, and is therefore a rational map. If h is not proper as a map from

$\mathbb{C} \setminus \{0,1\}$ to itself, then there is a sequence $\{x_n\} \subseteq \mathbb{C} \setminus \{0,1\}$ that converges to either $0, 1$ or ∞, such that some subsequence of the image sequence $\{h(x_n)\}$ converges to a finite point in $\mathbb{C} \setminus \{0,1\}$. Hence h is a rational map that misses at least one of the points $0, 1$ or ∞, and must therefore be constant.

On the other hand, assume that h is a proper map from $\mathbb{C} \setminus \{0,1\}$ to itself; then it is a non-constant rational map. Thus, if $h = \frac{P}{Q}$, where P and Q are two polynomials having no common factors, at least one of P or Q has to be non-constant. Also, note that h takes $\{0, 1, \infty\}$ to itself.

If P were non-linear, it would follow that either Q has the same degree as P, or Q is some constant C. In the latter case, both P and $P - Q$ are non-constant polynomials with disjoint zero sets. From this, it follows that P is either z^k or $(z-1)^k$, $k > 1$. Therefore, the equation $h = 1$ has roots different from 0 and 1, which is a contradiction. If P and Q have the same degree, then it follows that $\frac{P}{Q}$ is of the form R^k, where R is a non-constant rational function, and the value 1 is attained by k distinct values, which is also a contradiction. Hence P is linear, and a similar argument shows that Q is also linear. Hence h is a fractional linear transformation that takes $\{0, 1, \infty\}$ to itself. There are only six possibilities for the map h. From this it follows that, if for some z_0, $f_1(z_0, \cdot)$ is an automorphism, then $f_1(z, \cdot)$ is the same automorphism for all $z \in \mathbb{C} \setminus \{0,1\}$ (see Lemma 3.3.3). Together with the conclusion of the first paragraph, this proves that $f_1(z, \cdot)$ is either constant for all z, or is independent of z. Applying the same argument to f_2, we conclude that the conclusion of the Remmert–Stein theorem still holds.

Our choice of factors to consider in Theorem 3.1.1 motivated by the above example. The key fact used in the above example is that there are only finitely many proper holomorphic self-maps of $\mathbb{C} \setminus \{0,1\}$. This is not true for the domains $\mathbb{C}$ and $\mathbb{C} \setminus \{0\}$. But it is true for hyperbolic Riemann surfaces that are either compact, or compact but for finitely many punctures. This is the finiteness theorem of Imayoshi, which generalizes an earlier result due to de Franchis. We also used the big Picard theorem in the argument above. An analogue of the big Picard theorem for certain hyperbolic complex manifolds, given by Kobayashi, can be used in the situation of Theorem 3.1.1. The result of Imayoshi and the relevant generalization of the big Picard theorem, plus some technical necessities, are presented in Section 3.3.

If each factor in Theorem 3.1.1 is such that its boundary in the ambient compact Riemann surface is a non-empty indiscrete set, then the normal families argument used by Remmert and Stein can be adapted to deliver the conclusion of Theorem 3.1.1. We isolate this part of our proof as Proposition 3.4.1. The version of Montel's theorem that is needed for this proposition is presented in Section 3.2. The complete proof of Theorem 3.1.1 is presented in Section 3.4.

3.2 A version of Montel's theorem

In the proof of our main result, we need a version of Montel's theorem that is adapted to our situation. The proof of this version requires some general results about normal families. We state these results, with references, in this section. Throughout this section, M and N will denote complex manifolds, and $\mathcal{O}(M,N)$ will denote the space of holomorphic maps from M into N. We give $\mathcal{O}(M,N)$ the compact-open topology. We begin with the definition of a normal family.

Definition 3.2.1. A subset $\mathcal{F}$ of $\mathcal{O}(M,N)$ is said to be *normal* if every sequence of $\mathcal{F}$ contains a subsequence $\{f_n\}$ that is either convergent in $\mathcal{O}(M,N)$, or is compactly divergent. By the latter we mean that given compact sets $K \subseteq M$ and $H \subseteq N$, $f_n(K) \cap H = \emptyset$ for all sufficiently large n.

Result 3.2.2 (see [Kie70], Proposition 3)**.** *Let M be a complex manifold, and let N be a complete Kobayashi hyperbolic complex manifold. Then $\mathcal{O}(M,N)$ is a normal family.*

Result 3.2.3 (see [Kob67], Theorem 5.5)**.** *Let X be a hyperbolic Riemann surface. Then X is complete Kobayashi hyperbolic.*

We now state and prove the version of Montel's Theorem that we need, which is a corollary of the last two results.

Corollary 3.2.4. *Let X be a connected complex manifold and let R be a hyperbolic open connected subset of a compact Riemann surface S. Then, given any sequence $\{f_\nu\} \subset \mathcal{O}(X,R)$, there exists a subsequence $\{f_{\nu_k}\}$ and a holomorphic map $f_0 : X \to \overline{R}$ (the closure taken in S whenever R is non-compact) such that $f_{\nu_k} \to f_0$ uniformly on compact subsets of X.*

Proof. We begin by noting that if R is compact, then the result follows immediately from Results 3.2.3 and 3.2.2.

We now consider the case when R is a punctured Riemann surface. By Result 3.2.2, $\mathcal{O}(X,R)$ is a normal family. There is nothing to prove if there exists a subsequence $\{f_{\nu_k}\}$ that converges uniformly on compact subsets of X. Therefore, let us consider the case when we get only a compactly divergent subsequence $\{f_{\nu_k}\}$. Let $\{K_j : j \in \mathbb{Z}_+\}$ be an exhaustion of X by *connected* compact subsets, and let $\{L_j : j \in \mathbb{Z}_+\}$ be an exhaustion of R by compact subsets. Since R is obtained from S by deleting finitely many points from it, compact divergence implies that we can extract a further subsequence from $\{f_{\nu_k}\}$ — which we shall re-index again as $\{f_{\nu_k}\}$ — such that $f_{\nu_k}(K_1) \subset D^* \ \forall k$, where D^* is a deleted neighbourhood of one of the punctures, say p_0. Now, given any $j \in \mathbb{Z}_+$, there exists a $k(j) \in \mathbb{Z}_+$ such that, by the connectedness of the K_j's, we have:

$$f_{\nu_k}(K_j) \subset (D^* \setminus L_j) \ \ \forall k \geq k(j).$$

This just means that $f_{\nu_k} \to p_0$ uniformly on compacts as $k \to \infty$.

In the general case, as R is hyperbolic, we can make sufficiently many punctures in S to get a Riemann surface R' that is hyperbolic and $R \subseteq R' \subset S$. By considering each f_ν as a mapping in $\mathcal{O}(X, R')$, we can find, by the preceding argument, a subsequence $\{f_{\nu_k}\}$ and a holomorphic map $f_0 : X \to \overline{R'}$ such that $f_{\nu_k} \to f_0$ uniformly on compact subsets of X. As each $f_{\nu_k} \in \mathcal{O}(X, R)$, we must have $f_0 \in \mathcal{O}(X, \overline{R})$, and we are done. □

3.3 Some technical necessities

In this section we summarize several results that we need for the proof of Theorem 3.1.1. We state these results with appropriate references. We begin with an extension of a classical result due to de Franchis [dF13], which states that there are at most finitely many non-constant holomorphic mappings between two compact hyperbolic Riemann surfaces. We shall call a Riemann surface obtained by removing a finite, non-empty set of points from some compact Riemann surface a *punctured Riemann surface*. A Riemann surface obtained by removing n points from a compact Riemann surface of genus g will be called a *Riemann surface of finite type* (g, n). Imayoshi extended de Franchis' result as follows:

Result 3.3.1 (Imayoshi [Ima83])**.** *Let R be a Riemann surface of finite type and let S be a Riemann surface of finite type (g, n) with $2g - 2 + n > 0$. Then the set of non-constant holomorphic maps from R into S is at most finite.*

The above result combined with the following lemma will play a key role in the proof of the main theorem. To state this lemma, we need a definition.

Definition 3.3.2. Let $F : X \to Y$ be a map between two sets, and suppose that $X = X_1 \times \cdots \times X_n$. We say that F is *independent of X_j* if, for each fixed $(x_1^0, \ldots, x_{j-1}^0, x_{j+1}^0, \ldots, x_n^0)$, $x_i^0 \in X_i$, the map

$$X_j \ni x_j \longmapsto F(x_1^0, \ldots, x_{j-1}^0, x_j, x_{j+1}^0, \ldots, x_n^0),$$

is a constant map. We say that F *varies along X_j* if F is **not** independent of X_j.

Lemma 3.3.3. *Let R and S be as in Result 3.3.1, and let X be a connected complex manifold. Let $F : R \times X \to S$ be a holomorphic mapping with the property that for some $x_0 \in X$, the mapping $R \ni z \mapsto F(z, x_0) \in S$ is a non-constant mapping. Then F is independent of X.*

Proof. Let d_R and d_S be metrics that induce the topology of R and S, respectively. By Result 3.3.1, the set of non-constant holomorphic mappings from R to S is at most finite. By our hypotheses, there is at least one such map. Let $F_1, \ldots, F_k$ be the only distinct non-constant mappings in $\mathcal{O}(R, S)$. Let $x_0 \in X$ be such that the map $F(\cdot, x_0)$ is non-constant. By continuity of F, there is an X-open neighbourhood $U_0 \ni x_0$ such that $F(\cdot, x)$ is non-constant for $x \in U_0$. Choose $\varepsilon > 0$ and $r_{ij} \in R$, $1 \leq i, j \leq k$, $i \neq j$, such that $d_S(F_i(r_{ij}), F_j(r_{ij})) > \varepsilon$. By the continuity of F, we can find a neighbourhood $U \subset U_0$ of x_0 such that, for each of the r_{ij}'s, we have $d_S(F(r_{ij}, x), F(r_{ij}, y)) < \varepsilon \; \forall x, y \in U$. This is possible only if $F(\cdot, x) \equiv F(\cdot, y)$, $\forall x, y \in U$. It follows that

- $\exists j_0 \leq k$ such that $F(\cdot, x) = F_{j_0}\ \forall x \in U$;
- For each fixed $r \in R$, the map $F(r, \cdot)$ is constant on U.

As X is connected, the Identity Theorem implies that $F(r, \cdot) \equiv F_{j_0}(r)$. This proves that F is independent of X. □

The next result is the well known Remmert's Proper Mapping Theorem. For the proof, refer to [Chi89, p. 31].

Result 3.3.4 (Proper Mapping Theorem)**.** *Let X and Y be complex manifolds, and let A be an analytic subset of X. Let $f : A \to Y$ be a proper finite holomorphic map. Then, $f(A)$ is an analytic subset of Y, and at every $w \in f(A)$*

$$dim_w f(A) = \max\{dim_z A : f(z) = w\}.$$

In particular, $dim A = dim f(A)$. Furthermore, if $A = X$ and $dim(X) = dim(Y)$ then F is surjective.

The following result of Kobayashi [Kob98, p. 284] can be thought of as an higher dimensional analogue of the big Picard theorem. For this, we first need to make a definition.

Definition 3.3.5. Let Z be a complex manifold and let Y be a relatively compact complex submanifold of Z. We call a point $p \in \overline{Y}$ a *hyperbolic point* if every neighbourhood U of p contains a smaller neighbourhood V of p, $\overline{V} \subset U$, such that

$$K_Y(\overline{V} \cap Y, Y \setminus U) := \inf\{K_Y(x, y) : x \in \overline{V} \cap Y, y \in Y \setminus U\} > 0,$$

where K_Y denotes the Kobayashi pseudo-distance on Y. We say that Y is *hyperbolically imbedded* in Z if every point of $\overline{Y}$ is a hyperbolic point.

Result 3.3.6 (Kobayashi)**.** *Let Y and Z be complex manifolds, and let Y be hyperbolically imbedded in Z. Then every map $h \in \mathcal{O}(\mathbb{D}^*, Y)$ extends to a map $\tilde{h} \in \mathcal{O}(\mathbb{D}, Z)$.*

Lemma 3.3.7. *If Y is a hyperbolic open connected subset of a compact Riemann surface Z, then Y is hyperbolically imbedded in Z.*

Proof. The lemma is obvious if Y has only isolated boundary points. If not, then, as Y is hyperbolic, we can make sufficiently many punctures in Z to get a hyperbolic Riemann surface $\widetilde{Y}$ such that $Y \subset \widetilde{Y} \subset Z$. It follows that $\widetilde{Y}$ is hyperbolically imbedded in Z. Now let $y \in \overline{Y}$. Then, $y \in \overline{\widetilde{Y}}$. Let U be a neighbourhood of y, and let V be a smaller neighbourhood of y such that

$$K_{\widetilde{Y}}(\overline{V} \cap \widetilde{Y}, \widetilde{Y} \setminus U) > 0.$$

As $K_Y \geq K_{\widetilde{Y}}$ on $Y \times Y$ and $Y \subseteq \widetilde{Y}$, it follows that y is also a hyperbolic point of Y. Consequently, Y is hyperbolically imbedded in Z. □

We require one more result, a generalization of Rado's theorem that is proved in [Nar71].

Lemma 3.3.8. *Let* $(\phi_{\mu\nu})$, $1 \leq \mu \leq k$, $1 \leq \nu \leq l$, *be a matrix of holomorphic functions on* $D \subseteq U$, *where* D *and* U *are connected open subsets of* $\mathbb{C}$, *and* $U \setminus D$ *is a non-empty indiscrete set. Suppose that*

$$\prod_{\nu=1}^{l} \sum_{\mu=1}^{k} \left|\phi_{\mu\nu}(z)\right|^2 \to 0 \text{ as } D \ni z \to \zeta$$

for any $\zeta \in \partial D \cap U$. *Then, for some* ν_0, $1 \leq \nu_0 \leq l$, *we have*

$$\phi_{\mu\nu_0} \equiv 0,\ \mu = 1, \ldots, k.$$

Proof. Suppose each column of $(\phi_{\mu\nu})$ has a member that is not identically 0 on D. Let f be the product of these members. We extend f to be a function on U by defining $f \equiv 0$ on $U \setminus D$. By hypothesis, f is continuous on U and holomorphic on D. Therefore by the classical Rado's theorem $f \equiv 0$, a contradiction. □

3.4 Proof of the main theorem

We begin this section by considering a special case of Theorem 3.1.1 whose proof contains some technicalities. Since these technicalities would lengthen the proof of Theorem 3.1.1 if we were to embark on it directly, we shall isolate the technical portion of our proof in the following proposition. Its proof consists of rephrasing the Remmert–Stein argument relative to a coordinate patch; see [Nar71, pp. 71–78]. *We shall therefore be brief* and explain in detail only those points that differ from the proof in [Nar71].

Proposition 3.4.1. *Let* $X = X_1 \times \cdots \times X_n$ *and* $Y = Y_1 \times \cdots \times Y_n$, $n \geq 2$, *be complex manifolds. Assume that each* X_j *and each* Y_j *satisfy the hypothesis of Theorem 3.1.1 and that* Y *is non-compact. Further assume that, for each* j, $R_j \setminus X_j$ *is a non-empty indiscrete set. Let* $F : X \to Y$ *be a finite proper holomorphic map. Then, denoting* $z \in X_1 \times \cdots \times X_n$ *as* $(z_1, \ldots, z_n)$, *each* F_i *is of the form* $F_i(z_{\pi(i)})$, *where* π *is a permutation of* $\{1, \ldots, n\}$.

In particular, if there is a mapping with the above properties from X *to* Y, *then* Y *cannot have any compact factors.*

Proof. For $1 \leq j \leq n$, let R_j and S_j be as in Theorem 3.1.1. Let $p = (p_1, \ldots, p_n)$ be a point in $R_1 \times \cdots \times R_n$ such that, for each $1 \leq i \leq n$, p_i is a limit point of the set $R_i \setminus X_i$ and belongs to ∂X_i.

Let (U_k, ψ_k) be connected holomorphic co-ordinate charts of R_k, chosen in such a way that $p_i \in U_i$ and the image of $(\prod_{k=1}^{n} U_k) \cap X$ under each F_j lies in some holomorphic co-ordinate chart (V_j, ρ_j) of S_j. Let W_i be a connected component of $U_i \cap X_i$ such that $p_i \in \partial W_i$. For $(z_1, \ldots, z_n) \in \prod_{k=1}^{n} \psi_k(W_k)$, let

$$g_j(z_1, \ldots, z_n) := \rho_j \circ F_j(\psi_1^{-1}(z_1), \ldots, \psi_n^{-1}(z_n)).$$

In view of Corollary 3.2.4, we can rephrase the arguments in [Nar71, p. 75] to conclude:

$$\prod_{j=1}^{n}\sum_{k=1,k\neq i}^{n}\left|\frac{\partial g_j}{\partial z_k}(z_1,\dots,z_{i-1},w,\dots,z_n)\right|^2 \to 0 \text{ as } w\to\zeta\in\psi_i(\partial W_i),$$

where ζ is any arbitrary point in $\psi_i(\partial W_i)$. Let us take $D = \psi_i(W_i)$ and $U = \psi_i(U_i)$ in Lemma 3.3.8. Note that $\psi_i(p_i) \in U \setminus D$, whence $U \setminus D$ is indiscrete. Thus, we have that for each $(z_1,\dots, z_{i-1}, z_{i+1},\dots, z_n) \in \prod_{k=1,k\neq i}^{n}\psi_k(W_k)$, there is a $j = j(z)$ such that

$$h_j(z_1,\dots,z_{i-1},w,z_{i+1},\dots,z_n) := \sum_{k=1,k\neq i}^{n}\left|\frac{\partial g_j}{\partial z_k}(z_1,\dots,z_{i-1},w,\dots,z_n)\right|^2$$

is zero $\forall w \in D$. At this point, we can again argue exactly as in [Nar71, p. 75] to conclude that there exists an integer $\sigma(i), 1 \leq \sigma(i) \leq n$, such that

$$\frac{\partial g_{\sigma(i)}}{\partial z_k} \equiv 0 \text{ on } \psi_1(W_1)\times\cdots\times\psi_n(W_n),\ \ k=1,\dots,n,\ \ k\neq i.$$

Therefore on $W_1 \times \cdots \times W_n$, $F_{\sigma(i)}$ is independent of $z_1,\dots,z_{i-1},z_{i+1},\dots,z_n$. By applying the Identity Theorem, we conclude that $F_{\sigma(i)}$ is independent of the same variables on X. By Remmert's Proper Mapping Theorem, F is surjective. This implies that $F_{\sigma(i)}$ varies along X_i. Since the choice of $1 \leq i \leq n$ in the preceding argument was arbitrary, for each i there exists precisely one $\sigma(i)$ such that $F_{\sigma(i)}(z) = F_{\sigma(i)}(z_i)\ \forall z \in X$. The permutation $\pi = \sigma^{-1}$, and we are done with the proof of the first part.

To establish the final part of this result, assume that $Y_{s+1},\dots,Y_n$ are all compact, for some $s < n$. Fix an i as in the previous paragraph. The heart of the argument above, see [Nar71, p. 75], consists of using Montel's theorem (Corollary 3.2.4 in our present set-up) to construct a map $(\phi_1,\dots,\phi_n) : Z \to \partial Y$, where $Z := \prod_{k=1,k\neq i}^{n} X_k$. Set $E_j := \{z \in Z : \phi_j(z) \in \partial Y_j\}$. Clearly :

$$\{l : 1 \leq l \leq n,\ \mathrm{int}(E_l) \neq \emptyset\} \subseteq \{1,\dots,s\}. \tag{3.1}$$

In view of (3.1), the argument in [Nar71, p. 75] reveals that, for each $i,\sigma(i) \in \{1,\dots,s\}$. Since $s < n$, by assumption, there would exist $i \neq i'$ such that $\sigma(i) = \sigma(i')$. But this would contradict the surjectivity of F, and we are done.

□

The proof of Theorem 3.1.1. For $1 \leq j \leq n$, let R_j and S_j be the compact Riemann surfaces in the statement of the theorem. We start off with a simple consequence of the finiteness of F.

Claim A: *For any holomorphic finite map $F : X \to Y$, given any $X_i, 1 \leq i \leq n$, there is some F_j that varies along X_i.*

To see this, assume that there is a factor X_i such that all the F_j's are independent of X_i. Then for any point $x = (x_1,\dots,x_n) \in X$, by Definition 3.3.2, the inverse image of $F(x)$ contains the set $\{x_1\} \times \cdots \{x_{i-1}\} \times X_i \times \{x_{i+1}\} \times \cdots \{x_n\}$. But this contradicts the finiteness of F.

Let Y_C and Y_C denote the product of those factors of X and Y, respectively, that are either compact, or compact with finitely many punctures, and let X_B and Y_B denote the product of the remaining factors. Since Proposition 3.4.1 already establishes our theorem if $X_C = \emptyset$, we may assume, without loss of generality, that $X_C := X_1 \times \ldots \times X_p$, $1 \leq p \leq n$. Note that if $X_C = \emptyset$ and there exists a proper holomorphic map $F : X \to Y$, then Y cannot be compact.

Claim B: *The maps F_i are independent of $X_1, \ldots, X_p$, whenever $S_i \setminus Y_i$ is a non-empty indiscrete set.*

To see this, fix $x' \in X_2 \times \cdots \times X_n$. The map $F_i(\bullet, x')$ is a holomorphic map from X_1 into a hyperbolically imbedded Riemann surface. Now, R_1 is the compact Riemann surface from which X_1 is obtained by deleting at most finitely many points. From Result 3.3.6, it follows that $F_i(\bullet, x')$ extends holomorphically to a map $\widetilde{f_i}$ from R_1 into S_i. If $\widetilde{f_i}$ is non-constant, then by the compactness of R_1, it follows that the image set of R_1 under $\widetilde{f_i}$ is both compact and open. But, this means that $S_i = \widetilde{f_i}(R_1)$, which is not possible as $S_i \setminus Y_i$ is a non-empty indiscrete set, and $\widetilde{f_i}(R_1)$ is obtained by adjoining at most finitely many points to Y_i. This proves that F_i is independent of X_1. Repeating the same argument for the factors $X_2, \ldots, X_p$, the claim is proved.

Now note that, in view of Claim B, if $1 \leq i \leq p$ and F_j is a map that varies along X_i, then Y_j is either compact, or compact with finitely many punctures. Then, by Lemma 3.3.3, F_j is independent of *all the factors of X other than X_i*. Without loss of generality, we may assume that $Y_C = Y_1 \times \ldots Y_k$, $1 \leq k \leq n$. Combining our last deduction with Claim A, we infer that:

1. $p \leq k \leq n$;

2. Without loss of generality, there is an enumeration of the factors of Y_C such that for each $1 \leq i \leq p$, there is a unique $\sigma(i)$, $1 \leq \sigma(i) \leq p$, such that $F_{\sigma(i)}(z) = F_{\sigma(i)}(z_i)$ $\forall z \in X$.

Suppose $k > p$. Then, in view of the (harmless) assumption in (2), we need to analyze the behaviour of F_i, $p+1 \leq i \leq k$. Note that we already know from Claim B that $F_{k+1}, \ldots, F_n$ is independent of X_C. Assume that F_{p+1} varies along some $X_i, 1 \leq i \leq p$; then from Lemma 3.3.3, F_{p+1} is independent of all other factors of X. From Remmert's Proper Mapping Theorem (Result 3.3.4), F is a surjective map from X onto Y. Hence, combining the last two assertions with (2), $(F_1, \ldots, F_{p+1})$ determines a surjective holomorphic map $(F_1, \ldots, F_{p+1}) : X_C \to Y_1 \times \cdots \times Y_{p+1}$ from a space of dimension p to a space of dimension $p+1$, which contradicts Sard's theorem. Hence, F_{p+1} is independent of $X_1, \ldots, X_p$. Repeating the same argument for each map F_j, $p+1 \leq j \leq k$, we conclude that each F_j, $p+1 \leq j \leq n$, is independent of X_C.

Whether or not $k > p$, the previous paragraph implies that F_i, $p+1 \leq i \leq n$, are independent of X_C, whence they determine a surjective map $F_B = (F_{p+1}, \ldots, F_n) : X_B \to Y_{p+1} \times \ldots \times Y_n$. This map is clearly finite. We will now show that it is proper. Consider a compact set $K \subseteq Y_{p+1} \times \ldots \times Y_n$. We must show that $F_B^{-1}(K)$ is a compact subset of X_B. Let $H \subseteq Y_1 \times \cdots \times Y_p$ be some compact set. Then, by the properness of F, it follows that

$F^{-1}(H \times K)$ is compact. But, given the independence of the various F_i's from certain factors of X,

$$F^{-1}(H \times K) = (F_1, \ldots, F_p)^{-1}(H) \times F_B^{-1}(K).$$

Thus, $F_B^{-1}(K)$ is compact, as required.

As X_B is non-compact, and F_B is a proper map, it follows that $Y_{p+1} \times \cdots \times Y_n$ is also non-compact. We now apply Proposition 3.4.1 to the map F_B to get a permutation π of $\{p+1, \ldots, n\}$ such that, for each $p < i \leq n$, we have $F_i(z) = F_i(z_{\pi(i)})$. Juxtaposing π with the permutation σ^{-1} of $\{1, \ldots, p\}$, we are done. $\square$

4 Proper holomorphic self-maps of balanced domains

In this chapter, we prove that any proper holomorphic self-map of a smoothly bounded balanced pseudoconvex domain of finite type in $\mathbb{C}^n$, $n > 1$, is an automorphism. Here, the phrase "smoothly bounded" refers to a domain that is bounded and has a C^∞-smooth boundary. Alexander's theorem — i.e., Theorem 1.3.1 from the introduction — is thus a special case of this result. We first prove a proposition that gives the precise structure of the branch locus of such a proper holomorphic mapping, assuming that it is branched. The main novelty of our proof is the use of a recent result of Opshtein on the behaviour of the iterates of holomorphic self-maps of a certain class of domains. We use the aforementioned structure result, together with the finiteness of type, to deduce that the limit manifold for the iterates of a *branched* proper holomorphic mapping is necessarily a point or a Riemann surface. This contradicts Opshtein's theorem. A well-known result of Pinchuk delivers the proof.

4.1 Introduction and explanatory remarks

The central result of this chapter is:

Theorem 4.1.1. *Let $\Omega \subset \mathbb{C}^n$, $n > 1$, be a smoothly bounded pseudoconvex balanced domain of (D'Angelo) finite type. Then every proper holomorphic self mapping $F : \Omega \to \Omega$ is an automorphism.*

Let Ω and F be as in the above theorem. As discuseed in Section 1.5, a natural approach to proving a theorem such as Theorem 4.1.1 is to *assume* that F is branched, and to use this assumption together with the properties of Ω to reach a contradiction. To this end, we prove the following result that establishes an important property of the branch locus of F.

Proposition 4.1.2. *Let $\Omega \subset \mathbb{C}^n$, $n > 1$, be a smoothly bounded pseudoconvex balanced domain of (D'Angelo) finite type. Let $F : \Omega \to \Omega$ be a proper holomorphic mapping, and assume that the branch locus $V_F := \{z \in \Omega : \mathsf{Jac}_{\mathbb{C}}(F)(z) = 0\} \neq \emptyset$. Let X be an irreducible component of V_F. Then for each $z \in X$, the set $(\mathbb{C} \cdot z) \cap \Omega$ is contained in X.*

We ought to point out that Proposition 4.1.2 is a hypothetical statement. *If* F as in Theorem 4.1.1 were branched, *then* it would have the above structure. The thrust of our proof is that F can never be branched.

As hinted at in Section 1.5, an important object in our proof is a function $\tau : \partial\Omega \to \mathbb{Z}_+ \cup \{0\}$, which gives the tangential order of vanishing of the Levi determinant at each boundary point. This function was introduced by Bedford and Bell [BB82]. For the precise meanings of the phrases "Levi determinant" and "tangential order", we refer the reader to Section 4.2. Another key fact that is presented in that section is that, owing to the hypothesis that Ω is of (D'Angelo) finite type, τ is bounded on $\partial\Omega$.

Our proof of Theorem 4.1.1 may be summarized as follows:

- We begin by assuming that F is branched.
- If z_0 is a point in $F^{-1}\{0\}$ that is different from 0, it follows from Proposition 4.1.2 that $(\mathbb{C} \cdot z_0) \cap \overline{\Omega}$ is contained in a sequence of distinct irreducible components of the branch loci of the iterates F^k, $k = 1,2,3,\ldots$ We note that, by Lemma 2.2.5, F extends holomorphically to a neighbourhood of $\overline{\Omega}$, whence we may view these irreducible components as subvarieties of some neighbourhood of $\overline{\Omega}$.
- We pick a point $q \in (\mathbb{C} \cdot z_0) \cap \partial\Omega$. Each iterate of F must be branched at q. Using this fact, and that q is located on *distinct* irreducible components of the branch loci of F^k, one can show — using a result of Bell [Bel84a] — that τ must be unbounded on $\partial\Omega$. This is impossible, whence $F^{-1}\{0\} = \{0\}$.
- In particular, F fixes 0, whence there exists a limit manifold, call it M, associated to the iterates of F. The circular symmetry of Ω makes it possible to deduce that M is the intersection of a linear subspace of $\mathbb{C}^n$ with Ω, and that one may assume, without loss of generality, that $F|_M$ is given by

$$(z_1, \ldots, z_m, 0, \ldots, 0) \mapsto (e^{i\theta_1} z_1, \ldots, e^{i\theta_m} z_m, 0, \ldots, 0).$$

- If $m > 1$ then we can find a point $p \in \overline{M} \cap \partial\Omega$ that *also* lies in the (prolongation of) the branch locus of F. We examine the orbit of p under the action of the group generated by $F|_M$. The behaviour of the function τ along this orbit contradicts the upper semi-continuity of τ. Hence, $m \leq 1$, which, however, contradicts Opshtein's theorem: Result 4.3.7 below.
- This proves that our assumption that F is branched must be false. The desired conclusion now follows from a theorem of Pinchuk.

As discussed above, a result of Opshtein will play a fundamental role in the proof of Theorem 4.1.1. For this reason, we will need some definitions and facts from the theory of (iterative) dynamics of holomorphic self-maps of a domain in $\mathbb{C}^n$. These will be presented in Section 4.3. The proofs of Proposition 4.1.2 and Theorem 4.1.1 will be presented in Section 4.5.

Before, we proceed further, we clarify that, in this chapter, whenever we use use the word "smooth", it will refer to C^∞-smoothness unless specified otherwise.

4.2 Boundary geometry

In this section, we shall define the notions of pseudoconvexity and D'Angelo finite type. We shall then summarize the properties of the function τ, alluded to above, that is defined on the boundary of a smoothly bounded pseudoconvex domain. For an extensive treatment of the notion of finite type, refer to D'Angelo's book [D'A93]. We begin by defining the notion of Levi pseudoconvexity.

Definition 4.2.1. Let $D \subset \mathbb{C}^n$ be a bounded domain with smooth boundary, and let r be a defining function of D. We say that D is *Levi pseudoconvex* if for each $p \in \partial D$ we have:

$$\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \bar{z}_k}(p) v_j \bar{v}_k \geq 0 \quad \forall (v_1, \dots, v_n) \in T_p(\partial D) \cap iT_p(\partial D).$$

It is not hard to verify that the positivity condition in the above definition depends *only* on p and $v \in T_p(\partial D) \cap iT_p(\partial D)$, and not on the choice of the defining function r. We remark that $T_p(\partial D) \cap iT_p(\partial D)$ is the maximal complex subspace contained in $T_p(\partial D)$. Levi pseudoconvexity is the complex analogue of convexity and is preserved under biholomorphisms.

We now define the notion of D'Angelo finite type.

Definition 4.2.2. Let $D \subset \mathbb{C}$ be a domain, and let $f : D \to \mathbb{C}$ be a smooth function. We define the *multiplicity* of f at $p \in D$ to be the least positive integer k such that the homogeneous polynomial in $(z-p)$ and $(\overline{z}-\overline{p})$ of degree k in the Taylor expansion of $(f - f(p))$ around p is not identically zero, defining it to be $+\infty$ if no such k exists. The multiplicity of a $\mathbb{C}^n$-valued function is defined to be the minimum of the multiplicities of its components. We denote the multiplicity of a function f at a point p by $v_p(f)$.

Definition 4.2.3 (D'Angelo)**.** Let $M \subset \mathbb{C}^n$ be a smooth real hypersurface, and let $p \in M$. Let r be a defining function for M in some neighbourhood of the point p. We say that p is a point of *finite type* (also known as finite 1-type) if there is a constant $C > 0$ such that

$$\frac{v_0(r \circ \phi)}{v_0(\phi)} \leq C,$$

whenever $\phi : \mathbb{D} \to \mathbb{C}^n$ is a non-constant analytic disk such that $\phi(0) = p$.

As in the case of Definition 4.2.1 — and exactly for the same reasons — the above definition is independent of the choice of the defining function r. In the sequel, whenever we say that a domain is of finite type, we shall mean finite type in the sense of the above definition.

We now define a function τ on the boundary of any smoothly bounded pseudoconvex domain. This function was introduced by Bedford and Bell [BB82], and has been used successfully in the study of branching behaviour of proper holomorphic mappings. It is used multiple times in our proof of Theorem 4.1.1

Definition 4.2.4. Let $D \subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain and r a smooth defining function for D. Define

$$\Lambda_r := \det \begin{bmatrix} 0 & r_{z_i} \\ r_{\bar{z}_j} & r_{z_i \bar{z}_j} \end{bmatrix}_{i,j=1}^{n},$$

the *determinant of the Levi-form of* r (for a justification of this terminology, see [DK99, Section 2]). For $p \in \partial D$, we define $\tau(p)$ to be the smallest non-negative integer m such that there is a tangential differential operator T of order m on ∂D such that $T\Lambda_r(p) \neq 0$.

Remark 4.2.5. As any other defining function r' can be written as $h \cdot r$, where h is a positive smooth function defined on some neighbourhood U of the boundary point p, we see that the number $\tau(p)$ is independent of the choice of r. Note that by the pseudoconvexity of D, $\Lambda_r(p) \geq 0\ \forall p \in \partial D$, and $\Lambda_r(p) = 0$ if and only if p is a point of weak pseudoconvexity.

Remark 4.2.6. Note that τ is an upper semi-continuous function. To see this, observe that for $q \in \partial D$, the set $\{p \in \partial D : \tau(p) < \tau(q)\}$ is open, as its complement

$$\bigcap_T \{p \in \partial D : T\Lambda_r(p) = 0\},$$

where the intersection is over all tangential differential operators of order less than $\tau(q)$, is a closed set.

Suppose $f : D_1 \to D_2$ is a proper holomorphic mapping between bounded pseudoconvex domains with C^∞-smooth boundaries that extends smoothly to a ∂D_1-open neighbourhood of a point $p \in \partial D_1$. Let ρ be a defining function for D_2 such that $\rho \circ f$ is a local defining function for ∂D_1 near p (see [Bed84, Remark 2]). It follows that

$$\Lambda_{\rho \circ f}(z) = |\mathsf{Jac}_{\mathbb{C}}(f)(z)|^2 \Lambda_\rho(f(z)),$$

from which the next result is straightforward to prove.

Result 4.2.7 (Bell [Bel84a])**.** *Given a proper holomorphic mapping $g : D_1 \to D_2$ between bounded pseudoconvex domains in $\mathbb{C}^n$, $n > 1$, with smooth boundaries, if g extends smoothly to ∂D_1 in a neighbourhood of $p \in \partial D_1$, then $\tau(p) \geq \tau(g(p))$, and when $\tau(p) \neq \infty$, the following are equivalent :*

(*i*) $\tau(p) = \tau(g(p))$;

(*ii*) *g extends to a local diffeomorphism at p;*

(*iii*) $p \notin \overline{V}_g$.

Here, V_g denotes the branch locus, i.e., $V_g := \{z \in D_1 : \mathsf{Jac}_{\mathbb{C}}(g)(z) = 0\}$, of g.

Let us establish the following notation that we shall use in the remainder of this chapter. Given any open set $D \subset \mathbb{C}^n$ and a holomorphic map $f : D \to \mathbb{C}^n$, V_f will be defined as:

$$V_f := \{z \in D : \mathsf{Jac}_{\mathbb{C}}(f)(z) = 0\}.$$

The next two results give an idea as to why the τ function is relevant to our main theorem.

Let $D \subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain that is of finite type in the sense of D'Angelo and let $p = (p_1, \dots, p_n) \in \partial D$. From this, it can be argued that $\tau(p)$ is finite. As the D'Angelo 1-type of p is finite, if we pick a non-zero vector $(v_1, \dots, v_n) \in T_p(\partial D) \cap iT_p(\partial D)$, then there exists a $k \in \mathbb{Z}_+$ such that the homogeneous polynomial of degree k in the Taylor expansion of $r \circ \phi$ around $0 \in \mathbb{C}$ is not identically zero, where r here is a defining function of D and ϕ is the analytic disc

$$\phi : \zeta \longmapsto (p_1 + v_1\zeta, p_2 + v_2\zeta, \dots, p_n + v_n\zeta).$$

One can show from this that, for an appropriate choice of a vector $V^{ij} \in T_p(\partial D) \cap iT_p(\partial D)$, there is a tangential differential operator T^{ij} of order $\leq (k-2)$ such that T^{ij} applied to the (i, j)-th entry of the Levi matrix does *not* evaluate to 0 at p. However, finding a *single* finite-order tangential differential operator T such that $T\Lambda_r(p) \neq 0$ is quite technical. We could not find an elementary proof of Result 4.2.8 (see below) in the literature, although it has been made use of a number of times; see, for instance, [Pan91]. The main result of a recent work of Nicoara [Nic12, Main Theorem 1.1] provides an effective upper bound for τ in terms of the D'Angelo 1-type. For the purposes of this chapter, the following consequence of Nicoara's result suffices:

Result 4.2.8. *Let $D \subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain that is of finite type in the sense of D'Angelo. Then there is an $m \in \mathbb{Z}_+$ such that $\tau(p) \leq m \ \forall p \in \partial D$.*

The next result is by Coupet, Pan and Sukhov [CPS01]. The result, and its proof, is sufficiently important for our purposes that we provide a proof of it.

Lemma 4.2.9 ([CPS01])**.** *Let $\Omega \subset \mathbb{C}^n$, $n > 1$, be a bounded pseudoconvex domain with C^∞-smooth boundary that is also of finite type. Let $F : \Omega \to \Omega$ be a proper holomorphic mapping. Assume that F extends smoothly to $\partial\Omega$. If $V_F \neq \emptyset$, then for each $\nu \in \mathbb{Z}_+$, there exists an irreducible component L_ν of V_{F^ν} such that $L_i \neq L_j, i \neq j$, and*

$$L_{\nu+1} \subset F^{-1}(L_\nu) \ \ \forall \nu \in \mathbb{Z}_+.$$

Remark 4.2.10. Since Ω is pseudoconvex and of finite type, the assumption that F extends smoothly to $\partial\Omega$ is actually redundant, in view of the main result in [Cat87] together with [BC82] or [DF82]. However, even if these results had been unavailable, this assumption would not have been a hindrance owing to Lemma 2.2.5.

Proof. It is elementary to see that $V_{F^{\nu+1}} \supset F^{-1}(V_{F^\nu})$. Let L_1 be some irreducible component of V_F. Let L_2 be some irreducible component of $F^{-1}(L_1)$. Similarly, assume that $L_1, \ldots, L_k$ have been chosen such that L_j is some irreducible component of $F^{-1}(V_{F^{j-1}}), 1 \leq j \leq k$. Then, we may simply take L_{k+1} to be an irreducible component of $F^{-1}(L_k)$. Note that the restriction of F to each $L_{\nu+1}$ is proper, and consequently $F(L_{\nu+1}) = L_\nu$. We now show that *this procedure ensures that* $L_{\nu+1} \neq L_j \ \forall j \leq \nu, \nu = 1, 2, 3, \ldots$ Suppose not, and let m be the smallest positive integer such that $L_m = L_{m+p}$ for some positive integer p. If $m > 1$, then $F(L_m) = F(L_{m+p})$, and so $L_{m-1} = L_{m+p-1}$, contradicting the definition of m. So, $m = 1$ and $L_1 = L_{1+p}$. Since $F^p(L_{1+p}) = L_1$, we have $F^p(L_1) = L_1$. As $L_1 \subset V_{F^p}$, for any $q \in \overline{L}_1 \cap \partial\Omega$, we see, by Result 4.2.7, that

$$\cdots \tau(F^{kp}(q)) < \tau(F^{(k-1)p}(q)) < \cdots < \tau(q) < \infty. \tag{4.1}$$

This, in view of Result 4.2.8, contradicts the finite-type hypothesis on $\partial\Omega$.

□

4.3 Dynamics of holomorphic mappings

In this section we summarize some material from the theory of (iterative) dynamics of holomorphic self-maps of a taut manifold. A reference for the material in this section is [Aba89]. Given complex manifolds X and Y, $\mathrm{Hol}(X, Y)$ will denote the space of holomorphic mappings from X and Y, where the topology on $\mathrm{Hol}(X, Y)$ is the compact-open topology. We are interested in the set $\Gamma(f)$ which is defined to be the set of all limit points of the iterates of a holomorphic mapping $f \in \mathrm{Hol}(X, X)$, where X is a taut complex manifold. Of course, $\Gamma(f)$ might be empty. The following result describes the possible behaviours of the iterates.

Result 4.3.1 ([Aba89], Chapter 2.1)**.** *Let X be a taut manifold, and $f \in \mathrm{Hol}(X, X)$. Then either the sequence $\{f^k\}$ of iterates of f is compactly divergent, or there exists a complex submanifold M of X and a holomorphic retraction $\rho : X \to M$ (i.e., $\rho^2 = \rho$) such that every limit point $h \in \mathrm{Hol}(X, X)$ of $\{f^k\}$ is of the form $h = \gamma \circ \rho$, where γ is an automorphism of M. Moreover,*

1. *even ρ is a limit point of the sequence $\{f^k\}$,*
2. *$f(M) \subset M$, and $f|_M$ is an automorphism of M.*

Note that for a holomorphic retraction $\rho : X \to M$ as above, the fixed point set of ρ, $\mathsf{Fix}(\rho) = M$. The next result shows that the fixed point set of a holomorphic self-map of a bounded domain in $\mathbb{C}^n$ is a complex submanifold.

Result 4.3.2 (Vigué [Vig86, Vig90]). *Let D be a bounded domain in $\mathbb{C}^n$, $n \geq 2$, and let $f : D \to D$ be a holomorphic mapping. Then $\mathsf{Fix}(f)$ is a complex submanifold of D. If $a \in \mathsf{Fix}(f)$, its complex tangent space at a is given by*

$$\{v \in \mathbb{C}^n : f'(a)v = v\}$$

Definition 4.3.3. With the notation as in Result 4.3.1, we say that f is *non-recurrent* if the sequence $\{f^k\}$ of iterates of f is compactly divergent. Otherwise, we say that f is *recurrent*, and we call the map ρ the *limit retraction*, and the manifold M the *limit manifold*.

The behaviour of the iterates of a holomorphic self-map of a taut manifold X depends on whether f has a fixed point or not. The following theorem known as the Cartan-Carathéodory theorem gives a quantitative description of the behaviour of the differential f' at a fixed point of f; see [Aba89, Theorem 2.1.21] for a proof.

Result 4.3.4. *Let X be a taut complex manifold, and let $f \in \mathrm{Hol}(X, X)$ have some fixed point $z_0 \in X$. Then*

1. *the spectrum of $f'(z_0)$ is contained in $\overline{\mathbb{D}}$;*
2. $|\mathsf{Jac}_{\mathbb{C}}(f)(z_0)| = 1$ *if and only if f is an automorphism;*
3. *$T_{z_0}X$ admits a $f'(z_0)$-invariant splitting $T_{z_0}X = L_N \oplus L_U$ such that the spectrum of $f'(z_0)|_{L_N}$ is contained in $\mathbb{D}$, the spectrum of $f'(z_0)|_{L_U}$ is contained in ∂D and $f'(z_0)|_{L_U}$ is diagonalizable.*

The space L_U is called the *unitary space* of f at z_0. The above theorem can be used to give more information about the limit manifold of f at the fixed point; see [Aba89, Corollary 2.1.30] for a proof.

Result 4.3.5. *Let X be a taut manifold, and let $f \in \mathrm{Hol}(X, X)$ be such that $f(z_0) = z_0$ for some $z_0 \in X$. Then the unitary space of f at z_0 is the tangent space at z_0 of the limit manifold of f.*

The next result gives quite precise information about the set $\Gamma(f)$; see [Aba89, Corollary 2.4.4] for a proof.

Result 4.3.6. *Let X be a taut manifold, and let $f \in \mathrm{Hol}(X, X)$ be recurrent with limit retraction $\rho : X \to M$. Then $\Gamma(f)$ is isomorphic to a compact abelian subgroup of $\mathrm{Aut}(M)$, which is the closed subgroup generated by $f|_M \in \mathrm{Aut}(M)$.*

We point out that Result 4.3.1 guarantees that $f|_M$ is an automorphism of M.

We now state one of the main results in [Ops06]. This result is used in the final step in our proof of the main theorem. The result considers a slight generalization of a class of domains introduced by Sibony called B-regular domains. Refer to [Sib91b, Sib91a], and references therein, for a precise definition and related results.

Result 4.3.7 ([Ops06], Théorème A and Remarque 30)**.** *Let $D \subset \mathbb{C}^n$, $n \geq 2$ be a smoothly bounded pseudoconvex domain whose boundary is B-regular. Let $f : D \to D$ be a proper holomorphic self-map that is recurrent. Then the limit manifold of f is necessarily of dimension higher than* 1.

Remark 4.3.8. The notion of B-regularity is somewhat technical, whence we shall not define it here. The fact that is relevant to this chapter is that smoothly bounded pseudoconvex domains of finite type are B-regular. See [Sib91b, Section 2] for more on this matter.

Remark 4.3.9. In Opshtein's paper, [Ops06], Result 4.3.7 is actually stated for domains that admit a global smooth p.s.h. defining function at the boundary. For a proof that the result is also true for B-regular domains, refer to the Ph.D. thesis of Opshtein which, at the time of writing, is available at his homepage [Ops].

4.4 Some essential propositions

We begin with a simple application of the material in Section 2.3. This proposition has already been obtained by Vesentini in the more general context of reflexive Banach spaces. We use his techniques to give a simple proof in our special case.

Proposition 4.4.1. *Let $D \subset \mathbb{C}^n$ be a bounded, balanced pseudoconvex domain, all of whose boundary points are holomorphically extreme. Let $\rho : D \to D$ be a holomorphic retraction such that $\rho(0) = 0$. Then $M := \rho(D) = D \cap V$, where*

$$V := \{v \in \mathbb{C}^n : \rho'(0)v = v\}.$$

We remind the reader that we use the term "holomorphically extreme" here in the sense of Definition 2.3.2.

Proof. From Result 4.3.2, it follows that M is a connected complex submanifold of D whose complex tangent space at 0 is V. Let $v \in V \cap D$, $v \neq 0$. From Result 2.3.4 and our assumption on ∂D, it follows that the mapping

$$\phi : \mathbb{D} \ni \lambda \longmapsto \lambda v / M_D(v) \in D$$

is the unique (modulo $\mathrm{Aut}(\mathbb{D})$) κ_D-geodesic for $(0, v/M_D(v))$. Note that

$$\kappa_D(\rho \circ \phi(0); (\rho \circ \phi)'(0)) = \kappa_D(0; v/M_D(v)) = \kappa_D(\phi(0); v/M_D(v)),$$

whence $\rho \circ \phi$ is also a κ_D-geodesic for $(0, v/M_D(v))$. By uniqueness, it follows that $\rho \circ \phi = \phi \circ \psi$, where $\psi \in \mathrm{Aut}(\mathbb{D})$. But $\rho \circ \phi(0) = 0$, $(\rho \circ \phi)'(0) = v/M_D(v)$, and therefore by the Schwarz lemma, we must have $\rho \circ \phi = \phi$. In particular, $\phi(M_D(v)) = \rho \circ \phi(M_D(v))$, whence $v = \rho(v) \in M$. This proves that $V \cap D \subset M$. Note that V and M have the same complex dimension. Since, by Result 4.3.2, M is connected, it follows from the principle of analytic continuation that $M = D \cap V$. $\square$

We will now prove a lemma regarding analytic disks lying in a pseudoconvex domain of finite type. The boundary of such a domain cannot contain any non-trivial germs of analytic disks. The following Lemma shows that one can say more.

Lemma 4.4.2. *Let $D \subset \mathbb{C}^n$, $n \geq 2$, be a smoothly bounded pseudoconvex domain of finite type. Let $\psi : \mathbb{D} \to \overline{D}$ be a holomorphic map such that $\psi(\mathbb{D}) \cap \partial D \neq \emptyset$. Then ψ is constant*

Proof. Let $p \in \psi(\mathbb{D}) \cap \partial D$. Let us fix a defining function ρ for $D, \rho \in C^\infty(U)$, where U is a neighbourhood of $\overline{D}$, such that $|\nabla\rho(w)| = 1\ \forall w \in \partial D$. Let us write, for any $\varepsilon, r > 0$:

$$\hat{n} := \nabla\rho(p), \text{ the unit outward normal to } \partial D \text{ at } p,$$
$$W(r,\varepsilon) := B(p,r) \cap \{w \in D : \rho(w) > -\varepsilon\}.$$

Let r_0 and ε_0 be so small that $\forall w \in W(r_0,\varepsilon_0)$, $\forall \varepsilon \in (0,\varepsilon_0)$, the line segment $[w-\varepsilon\hat{n}, w] \subset U$, and $\rho^{-1}\{-\varepsilon\}$ is a hypersurface. By Taylor's theorem:

$$\rho(w - \varepsilon\hat{n}) = -\varepsilon\nabla\rho(w) \cdot \hat{n} + O(\varepsilon^2)\ \ \forall w \in W(r_0,\varepsilon_0),\ \forall \varepsilon \in (0,\varepsilon_0), \tag{4.2}$$

where, for vectors A, B, $A \cdot B$ denotes the standard inner product on $\mathbb{R}^{2n}$. We can assume that $\varepsilon_0, r_0 > 0$ are so chosen that $\rho(w) \cdot \hat{n} > 1/2\ \forall w \in W(r_0,\varepsilon_0)$. By (4.2), we can find an $\varepsilon_1 \in (0,\varepsilon_0)$ so small that

$$\rho(w - \varepsilon\hat{n}) < 0 \quad \forall w \in W(r_0,\varepsilon_1),\ \forall \varepsilon \in (0,\varepsilon_1). \tag{4.3}$$

Let $G := W(r_0,\varepsilon_1)$. Clearly, by (4.3), $G - t\hat{n} \subset D\ \forall t \in (0,\varepsilon_1)$.

Let $\zeta \in \psi^{-1}\{p\}$. There is an open neighbourhood N of ζ, $N \subset \mathbb{D}$, such that $\psi(N) \subset G \cup \partial D$. Now, define the maps $\phi_t \in \mathcal{O}(N; D)$ by

$$\phi_t(z) := \psi|_N(z) - t\hat{n},\ \ t \in (0,\varepsilon_1);$$

it is by the conclusion of the argument in the last paragraph that we see that $\phi_t(N) \subset D\ \ \forall t \in (0,\varepsilon_1)$. By construction

$$\phi_t \longrightarrow \psi|_N \ \text{ uniformly as } t \to 0^+.$$

Now, as D is smoothly bounded and pseudoconvex, it is taut, whence, as $p \in \psi(N) \cap \partial D$, it follows that $\psi(N) \subset \partial D$. As D is of finite type, it cannot contain any germs of analytic varieties of positive dimension. Hence, $\psi|_N$ is a constant, whence the result. □

4.5 Proofs of Proposition 4.1.2 and Theorem 4.1.1

If we assume that the map $F : \Omega \to \Omega$, as stated in Theorem 4.1.1 is branched, then our assumptions on the geometry of Ω gives us a structural result for the branch locus V_F of F. We begin with the proof of this result — i.e., Proposition 4.1.2.

A comment about notation: in what follows, $\dim_H(S)$ will denote the Hausdorff dimension of the set $S \subset \mathbb{C}^n$.

The proof of Proposition 4.1.2. Let $X_1, \dots, X_m$ be the distinct irreducible components of the variety V_F. By Lemma 2.2.5, F extends holomorphically to a neighbourhood N of $\overline{\Omega}$. For the moment, let $\widetilde{F}$ denote this extension. It is possible that there exists an irreducible component Y of $\mathsf{Jac}_{\mathbb{C}}(\widetilde{F})^{-1}\{0\}$ such that $X_j, X_k \subset Y$, $j \neq k$. However, it is a basic fact that there exists a domain $\Omega' \Supset \Omega$ such that the irreducible components of $(\mathsf{Jac}_{\mathbb{C}}(\widetilde{F}))^{-1}\{0\}$ are in one-to-one correspondence with $\{X_1, \dots, X_m\}$. For this reason, we shall not use different symbols for $F : \Omega \to \Omega$ and its extension $\widetilde{F}$, or for $X_1, \dots, X_M$ and their respective prolongations across $\partial\Omega$. We first need the following lemma:

Lemma 4.5.1. *Let X be an arbitrary irreducible component of the variety V_F, viewed as a subvariety of a domain $\Omega' \Supset \Omega$ that has the property that the irreducible components of V_F are in one-to-one correspondence with those of* $(\mathsf{Jac}_{\mathbb{C}}(F)|_{\Omega})^{-1}\{0\}$. *Let $E := \partial\Omega \cap X$. There exists a closed, nowhere dense (relative to E) subset $\mathcal{E} \subset E$ such that, for each $p \in E \setminus \mathcal{E}$, there exists a connected neighbourhood $N_p \ni p$ such that*

$$\dim_H(E \cap N_p) \geq 2n - 3.$$

Proof. Let S denote the subvariety of singular points of X and set

$$\mathcal{E} := \partial\Omega \cap S.$$

Pick a point $p \in E \setminus \mathcal{E}$. By definition, $\exists r_p > 0$ such that $\overline{B}(p, r_p) \cap \mathcal{E} = \emptyset$ and $X \cap B(p, r_p)$ is a complex submanifold of $B(p, r_p)$.

Claim: *For each $r \in (0, r_p)$, $(X \setminus \overline{\Omega}) \cap B(p, r) \neq \emptyset$.*

Assume this is false. Then, $\exists r \in (0, r_p)$ such that $X \cap B(p, r) \subset \overline{\Omega}$. This implies that there exists a non-constant holomorphic map $\psi : \mathbb{D} \to \mathbb{C}^n$ such that $\psi(\mathbb{D}) \subset X \cap B(p, r) \cap \overline{\Omega}$ and $\psi(\mathbb{D}) \cap \partial\Omega \neq \emptyset$. But this is impossible by Lemma 4.4.2. Hence the claim.

Now, let $r_p^* \in (0, r_p)$ be so so small that $B(p, r_p^*) \setminus \partial\Omega$ has exactly two connected components (possible as $\partial\Omega$ is an imbedded smooth submanifold),

$$B(p, r_p^*) \setminus \partial\Omega = C^+ \sqcup C^-.$$

By our above claim, $C^\pm \cap X \neq \emptyset$. Thus $\partial\Omega \cap X \cap B(p, r_p^*)$ disconnects the manifold $X \cap B(p, r_p^*)$.

In what follows, $\dim_I$ will denote the *inductive dimension*. The precise definition is rather involved and we refer the reader to [HW41, Chapters II and III]. The fact that we need is Corollary 1 to [HW41, Theorem IV.4]: since $\partial\Omega \cap X \cap B(p, r_p^*)$ disconnects $X \cap B(p, r_p^*)$

$$\dim_I(\partial\Omega \cap X \cap B(p, r_p^*)) \geq \dim_{\mathbb{R}}(X \cap B(p, r_p^*)) - 1 = 2n - 3. \tag{4.4}$$

It is well-known that the Hausdorff dimension dominates the inductive dimension; see for instance [HW41, Chapter VII, §4]. By (4.4), therefore, writing $N_p := B(p, r_p^*)$,

$$\dim_H(N_p \cap E) \geq 2n - 3.$$

The set $\mathcal{E}$ has all the properties stated in the lemma, and we are done. □

Let us fix a point $z_0 \in E$ for the moment. From Result 4.2.7, all the points of E are necessarily weakly pseudoconvex. Note that $\tau(e^{i\theta}z_0) = \tau(z_0)$. From the fact that τ is upper semi-continuous, it follows that the set $\{w \in \partial\Omega : \tau(w) < \tau(F(z_0)) + 1\}$ is open in $\partial\Omega$, and consequently so is its inverse image under F, $\{z \in \partial\Omega : \tau(F(z)) < \tau(F(z_0)) + 1\}$. The latter set obviously contains z_0. This implies that for θ close to 0, we must have $\tau(F(e^{i\theta}z_0)) \leq \tau(F(z_0)) < \tau(z_0) = \tau(e^{i\theta}z_0)$ which, by Result 4.2.7, implies that for θ close to 0, we have $e^{i\theta}z_0 \in V_F$. Restricting $\mathsf{Jac}_{\mathbb{C}}(F)$ to the set $\mathbb{D} \cdot z_0$, and observing that the boundary-values of this restriction vanishes on an arc of $\partial\mathbb{D}$, we see that $\mathsf{Jac}_{\mathbb{C}}(F)$ must vanish on the set $\mathbb{D} \cdot z_0$. As $z_0 \in E$ was arbitrary, we get that for each $z \in E$, $\mathbb{D} \cdot z \subset X_i$, for some i. Let us define

$$E_i := \{w \in E : \mathbb{D} \cdot w \subset X_i\},\ i = 1, \ldots, m.$$

Since, by Lemma 4.5.1, E is of Hausdorff dimension at least $2n-3$, there is an $i_0, 1 \leq i_0 \leq m$, such that $\dim_H(E_{i_0}) \geq 2n-3$. Let us call this set E'. Then:

$$\bigcup_{z \in E'} \mathbb{D} \cdot z \subset X_{i_0}.$$

As X and X_{i_0} are irreducible varieties whose intersection is of Hausdorff dimension at least $2n-3$, it must be that $X = X_{i_0}$.

We have proved that

$$\bigcup_{z \in E'} \mathbb{D} \cdot z \subset X. \tag{4.5}$$

Now fix $\lambda \in \mathbb{D}$, and consider the holomorphic function $h_\lambda(z) := \mathsf{Jac}_{\mathbb{C}}(F)(\lambda z)$ defined on X. From what we have shown, (4.5) in particular, h_λ vanishes on a subset of Hausdorff dimension at least $2n-3$ of the irreducible variety X. Hence h_λ must vanish identically on X, and this is true for each $\lambda \in \mathbb{D}$. Thus, we have shown that, given $z \in X$, $(\mathbb{D} \cdot z) \cap \Omega \subset V_F$. Since V_F comprises finitely many irreducible components, by a similar argument as in the previous paragraph (with the role of E now taken by X), we actually have $(\mathbb{D} \cdot z) \cap \Omega \subset X$. By analytic continuation, it follows that if $z \in X$, then $(\mathbb{C} \cdot z) \cap \Omega \subset X$. □

We now have all the tools needed to prove our main theorem.

The proof of Theorem 4.1.1. By Theorem 1.3.3, it suffices to prove that F is unbranched. So we will assume that F is branched and reach a contradiction. We will not, hereafter, remark upon the well-definedness of quantities such as $\mathsf{Jac}_{\mathbb{C}}(F)(p)$ for $p \in \partial\Omega$. In view of Lemma 2.2.5, this is indeed well-defined.

Step 1. *Proving that* $F^{-1}\{0\} = \{0\}$. If not, there is point $0 \neq z_0 \in \Omega$ such that $F(z_0) = 0$. From Lemma 4.2.9, we have distinct irreducible varieties $L_i \subset V_{F^i}$ such that $F|_{L_{i+1}} : L_{i+1} \to L_i$ is a proper holomorphic mapping. Moreover, from the manner in which we select the L_i's in the proof of that lemma and from the structure of the L_i's as given by Proposition 4.1.2, it is clear that we can select the L_i's in such a manner that $z_0 \in L_i\ \forall i > 1$. To be specific:

having found $L_1, \dots, L_i$ with the last property, it follows from Proposition 4.1.2 that z_o would belong to *some* irreducible component of $F^{-1}(L_i)$, which we choose as L_{i+1}. Again from Proposition 4.1.2, it follows that $\Lambda := (\mathbb{C} \cdot z_0) \cap \Omega \subset L_i \; \forall i > 1$. This means that the sets $F^k(\Lambda) \subset L_2, \forall k \in \mathbb{Z}_+$. Let $q \in (\mathbb{C} \cdot z_0) \cap \Omega = \overline{\Lambda} \setminus \Lambda$. It is elementary that

$$V_{F^n} = \bigcup_{k=0}^{n-1} (F^k)^{-1}(V_F), \quad n \in \mathbb{Z}_+,$$

with the understanding that $F^0 = \mathsf{id}_\Omega$. Thus — we refer to the recipe for the L_i's in Lemma 4.2.9 — $L_2 \subset V_{F^{2k}} \forall k \in \mathbb{Z}_+$. Note that $q \in \overline{L_2} \setminus L_2$. At this stage, we are precisely in the situation prior to (4.1) in the proof of Lemma 4.2.9, except that q belongs to (the prolongation of) the branch locus of F^2. Therefore, it follows as in the proof of Lemma 4.2.9 (taking $p = 2$ in the relevant argument), that

$$\cdots < \tau(F^{2k}(q)) < \cdots < \tau(F^4(q)) < \tau(F^2(q)) < \tau(q) < \infty,$$

which contradicts the fact that Ω is pseudoconvex and of finite type (Result 4.2.8). Our claim follows.

We note, though we shall not make use of it in what follows, that by Theorem 1.2.9, it follows that F is a polynomial mapping.

Step 2. *Analyzing F on its limit manifold.* As 0 is a fixed point of F, it follows that F is recurrent. Let $\rho : \Omega \to M$ be the limit retraction. As Ω is pseudoconvex and of finite type, it follows from Lemma 4.4.2 that every point in $\partial\Omega$ is holomorphically extreme, and consequently from Proposition 4.4.1 and Result 4.3.5, it follows that $M = L_U \cap \Omega$, where L_U is the unitary space of F at 0. Recall that $F'(0)|_{L_U}$ is diagonalizable; see Result 4.3.5. Thus, without loss of generality (replacing Ω by a suitable linear image and conjugating F by a suitable linear operator, if necessary), we may assume that $L_U = \mathbb{C}^m \times \{0_{\mathbb{C}^{n-m}}\}$, and that $F'(0)|_{L_U}$ is given by

$$(z_1, z_2, \dots, z_m, 0, \dots, 0) \mapsto (e^{i\theta_1} z_1, e^{i\theta_2} z_2, \dots, e^{i\theta_m} z_m, 0, \dots, 0). \tag{4.6}$$

By Cartan's uniqueness theorem, it also follows that $F|_M \equiv F'(0)|_M$.

Step 3. *Proving that* $\dim M \leq 1$. Suppose $\dim M > 1$. From the previous steps, we have that $0 \in V_F \cap M$. Therefore the set $\{z \in M : \mathsf{Jac}_{\mathbb{C}}(F)(z) = 0\}$ is a non-empty analytic subvariety of M. As $\dim M > 1$, it follows that there exists a point $p \in \overline{M} \cap \partial\Omega$ such that $\mathsf{Jac}_{\mathbb{C}}(F)(p) = 0$.

From Result 4.3.6, the set $\Gamma(F)$ comprising of the various limit points of F in $\mathrm{Hol}(\Omega, \Omega)$ is isomorphic to a compact abelian subgroup of $\mathrm{Aut}(M)$, and in fact

$$\Gamma(F) \cong \overline{\{F^k|_M : k \in \mathbb{Z}\}}.$$

This means that there is a strictly increasing sequence $\{n_k\} \subset \mathbb{N}$ such that $F^{n^k}|_M \to F^{-1}|_M$ in $\mathrm{Aut}(M)$. But the maps $F^k|_M, k \in \mathbb{Z}$, are all of the special form (4.6), and consequently $F^{n_k}|_{\overline{M}} \to F^{-1}|_{\overline{M}}$ uniformly on $\overline{M}$. Therefore, we must have $F^{-n_k}|_{\overline{M}} \to F|_{\overline{M}}$ uniformly on $\overline{M}$. Let $p_k := F^{-n_k}|_{\overline{M}}(p)$. By Result 4.2.7, it follows that $\tau(p_{k+1}) \geq \tau(p_k) \geq \tau(p)$. But, $p_k \to F(p)$, and as $p \in \overline{V}_F, \tau(F(p)) < \tau(p)$, which means that

$$\limsup_{k \to \infty} \tau(p_k) \geq \tau(p) > \tau(F(p)),$$

which contradicts the fact that τ is upper semi-continuous. This proves that $\dim(M) \leq 1$.

The conclusion of Step 3 is in conflict with the conclusion of Opshtein's theorem, i.e., Result 4.3.7. Therefore, F is unbranched, and Theorem 1.3.3 proves that F is an automorphism. □

5 Proper holomorphic maps between bounded symmetric domains

In this chapter, we prove that a proper holomorphic map between two non-planar bounded symmetric domains of the same dimension, one of them being irreducible, is a biholomorphism. Our methods allow us to give a single, all-encompassing argument that unifies the various special cases in which this result is known. We discuss an application of these methods to domains having non-compact automorphism groups that are not assumed to act transitively.

Most of the material presented below also appears in the paper [BJ13].

5.1 Introduction and explanatory remarks

The main result of this chapter is the following:

Theorem 5.1.1. *Let D_1 and D_2 be two bounded symmetric domains of complex dimension $n \geq 2$. Assume that either D_1 or D_2 is irreducible. Then, any proper holomorphic mapping of D_1 into D_2 is a biholomorphism.*

As we had mentioned briefly in Section 1.4, our proof relies on the fact that any bounded symmetric domain $D \subset \mathbb{C}^n$ has a special realization (i.e., the Harish-Chandra realization) as the unit ball of $\mathbb{C}^n$ under some $\mathbb{C}$-norm . This special realization allows us to:

a) Define a triple product on $\mathbb{C}^n$ that makes $\mathbb{C}^n$ into a Jordan triple system, which allows us to give a description of the boundary geometry of a bounded symmetric domain along the lines alluded to in Section 1.5;

b) Adapt some aspects of Alexander's proof: specifically, those aspects that rely on the unit Euclidean ball being a convex balanced domain.

Before proceeding further, we point out that the Bergman metric associated to a bounded symmetric domain is geodesically complete. From this it follows easily that a bounded symmetric domain is homogeneous: the symmetry associated to the mid-point of a geodesic joining two given points interchanges the two points.

We will now present a brief outline of the proof of Theorem 5.1.1. An important lemma used in our proof is the Key Lemma below, which is reminiscent of a lemma of Rudin used by him to give a somewhat different proof ([Rud08, Chapter 15]) of Alexander's theorem.

Key Lemma 5.1.2. *Let D be a realization of an irreducible bounded symmetric domain of dimension $n \geq 2$ as a bounded convex balanced domain in $\mathbb{C}^n$. For $z \in D \setminus \{0\}$, let $\Delta_z := \{\zeta z : \zeta \in \mathbb{C}$ and $\zeta z \in D\}$. Let W_1 and W_2 be two regions in D such that $0 \in W_1 \cap W_2$ and let $F : D \to D$ be a holomorphic map. Assume that:*

- *(i) F maps W_1 biholomorphically onto W_2 with $F(0) = 0$.*
- *(ii) There exists a non-empty open set $U \subset W_1 \setminus \{0\}$ such that, for each $z \in U$, $\Delta_z \subset W_1$ and $\Delta_{F(z)} \subset W_2$.*

Then, F is an automorphism of D.

This is a consequence of Vigué's Schwarz lemma [Vig91] (Result 5.4.4 below), and the irreducibility of D is essential to this lemma. The definition of irreducibility is a *negative* one and is not very useful, without the use of any additional machinery, for proving results about irreducible domains. The machinery of Jordan triple systems gives us a convenient description of the boundary of an irreducible bounded symmetric domain — see Result 5.3.3 below — which is used crucially in the proof of Vigué's Schwarz lemma.

Our proof of Theorem 5.1.1 may be summarized as follows (we will assume here that D_1 and D_2 are Harish-Chandra realizations of the domains in question):

- By a consequence of a result of Bell (Lemma 2.2.5), F extends to a neighbourhood of $\overline{D}_1$ and we can find a point p in the Shilov boundary of D_1, and a small ball B around it, such that $F|_B$ is a biholomorphism.
- We may assume that $F(0) = 0$. Let $\{a_k\}$ be a sequence in $D_1 \cap B$ converging to p and let $b_k := F(a_k)$. Let $\phi_k^j \in \mathrm{Aut}(D_j)$ be an automorphism that maps 0 to a_k if $j = 1$, and to b_k if $j = 2$. It turns out that both p and $F(p)$ are peak points, whence $\phi_k^j \longrightarrow p^{(j)}$ uniformly on compact subsets, where $p^{(1)} := p$ and $p^{(2)} := F(p)$.
- Using the Schwarz lemma for convex balanced domains (Result 5.4.3 below) we show that a subsequence of $\{(\phi_j^2)^{-1} \circ F \circ \phi_j^1\}$ converges to a linear map and that, owing to the tautness of D_1 and D_2, this map is a biholomorphism of D_1 onto D_2.
- We may now take $D_1 = D_2 = D$. We shall use Lemma 5.1.2 with $W_1 = (\phi_k^1)^{-1}(D \cap B)$ and $W_2 = (\phi_k^2)^{-1}(D \cap F(B))$ for k sufficiently large.
- Since the analytic discs Δ_z and $\Delta_{F(z)}$ are not relatively compact in D, the mode of convergence of $\{\phi_k^j\}$ isn't *a priori* good enough to infer that appropriate families of these discs will be swallowed up by W_j, $j = 1, 2$. By Bell's theorem, each ϕ_k^j extends

to some neighbourhood of D. As we had mentioned at the end of Section 1.5, we have expressions for ϕ_k^j that are explicit enough to give estimates on ϕ_j^k that are independent of k. This estimate can be used to show that $\{\phi_k^j\}$, passing to a subsequence and relabelling, if necessary, converges *uniformly* on certain special special circular subsets of D that are adherent to ∂D. This is enough to overcome the difficulty just described.

We now come to the second main result of this chapter, obtained in collaboration with Gautam Bharali. The methods outlined above have the advantage that they do not rely on the fine structure of bounded symmetric domains. This allowed us to generalize some of the methods outlined above assuming only that the automorphism group of D_1 is non-compact. We do, however, assume that D_1 is convex and balanced. One can, in fact, prove a version of Theorem 5.1.3 that requires D_1 merely to be Kobayashi hyperbolic. However, in this case, the biholomorphism of D_1 onto D_2 will not, in general, be linear. We prefer the version below: the conclusion that there exists a *linear* equivalence places Theorem 5.1.3 among the rigidity theorems alluded to in Remark 1.5.9.

Theorem 5.1.3. *Let D_1 be a bounded convex balanced domain in $\mathbb{C}^n$ whose automorphism group is non-compact and let p be a boundary orbit-accumulation point. Let D_2 be a realization of a bounded symmetric domain as a bounded convex balanced domain in $\mathbb{C}^n$. Assume that there is a neighbourhood U of p and a biholomorphic map $F : U \to \mathbb{C}^n$ such that $F(U \cap D_1) \subset D_2$ and $F(U \cap \partial D_1) \subset \partial D_2$. Assume that either p or $F(p)$ is a peak point. Then, there exists a* linear *map that maps D_1 biholomorphically onto D_2.*

The layout of this chapter is as follows. Since Jordan triple systems play a vital role in describing not just the structure of the boundary of a bounded symmetric domain, but also some of its key automorphisms, we begin with a primer on Jordan triple systems. Readers who are familiar with Jordan triple systems can skip to Section 5.3, where we discuss the boundary geometry of bounded symmetric domains. Section 5.4 is devoted to stating and proving certain propositions that are essential to our proofs. Finally, in Sections 5.5 and 5.6, we present the proofs of the results stated above

5.2 A primer on Jordan triple systems

There is a natural connection between bounded symmetric domains and certain Hermitian Jordan triple systems. This section collects several definitions and results that are required to give a coherent description of the boundary of a bounded symmetric domain (which we shall discuss in the next section).

Unless otherwise stated, the results in this section can be found in the UC-Irvine lectures by Loos [Loo77] describing how Jordan triple systems can be used to study the geometry of bounded symmetric domains.

Definition 5.2.1. A *Hermitian Jordan triple system* is a complex vector space V endowed with a triple product $(x, y, z) \longmapsto \{x, y, z\}$ that is symmetric and bilinear in x and z and conjugate-linear in y, and satisfies the Jordan identity

$$\begin{aligned}\{x, y, \{u, v, w\}\} &- \{u, v, \{x, y, w\}\} \\ &= \{\{x, y, u\}, v, w\} - \{u, \{y, x, v\}, w\} \;\; \forall x, y, u, v, w \in V.\end{aligned}$$

Such a system is said to be *positive* if for each $x \in V \setminus \{0\}$ for which $\{x, x, x\} = \lambda x$ (where λ is a scalar), we have $\lambda > 0$.

As mentioned in Section 1.4, a bounded symmetric domain of complex dimension n has a realization D as a bounded convex balanced domain in $\mathbb{C}^n$. Let $(z, \dots, z_n)$ be the global holomorphic coordinates coming from the product structure on $\mathbb{C}^n$ and let $(\epsilon_1, \dots, \epsilon_n)$ denote the standard ordered basis of $\mathbb{C}^n$. Let K_D denote the Bergman kernel of (the above realization of) D and h_D the Bergman metric at 0. The function $\{\cdot, \cdot, \cdot\} : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ obtained by the requirement

$$h_D(\{\epsilon_i, \epsilon_j, \epsilon_k\}, \epsilon_l) = \left.\frac{\partial^4 \log K_D(z, z)}{\partial z_i \partial \overline{z}_j \partial z_k \partial \overline{z}_l}\right|_{z=0}, \tag{5.1}$$

and by extending $\mathbb{C}$-linearly in the first and third variables and $\mathbb{C}$-antilinearly in the second, has the property that $(\mathbb{C}^n, \{\cdot, \cdot, \cdot\})$ is a positive Hermitian Jordan triple system (abbreviated hereafter as PHJTS). This relationship is a one-to-one correspondence between finite-dimensional PHJTSs and bounded symmetric domains — which we shall make more precise in Section 5.3.

Let $(V, \{\cdot, \cdot, \cdot\})$ be a HJTS. It will be convenient to work with the operators

$$\mathbf{D}(x, y)z = \mathbf{Q}(x, z)y := \{x, y, z\}. \tag{5.2}$$

We define the operator $Q : V \to \mathrm{End}(V)$ by $Q(x)y := \mathbf{Q}(x, x)y/2$. For any $x \in V$, we can define the so-called *odd powers* of x recursively by:

$$x^{(1)} := x \quad \text{and} \quad x^{(2p+1)} := Q(x)x^{(2p-1)} \text{ if } p \geq 1.$$

A vector $e \in V$ is called a tripotent if $e^{(3)} = e$.

Tripotents are important to this discussion because:

1. A finite-dimensional PHJTS has plenty of non-zero tripotents.
2. Given a finite-dimensional PHJTS $(V, \{\cdot, \cdot, \cdot\})$, any vector V has a certain canonical decomposition as a linear combination of tripotents.
3. In a finite-dimensional PHJTS, the set of tripotents forms a real-analytic submanifold.

We refer the interested reader to [Loo77, Chapter 3] for details of the first fact. As for the second fact, we need a couple of new notions. First: given a HJTS $(V, \{\cdot,\cdot,\cdot\})$, we say that two tripotents $e_1, e_2 \in V$ are orthogonal if $\mathsf{D}(e_1, e_2) = 0$. Second: given $x \in V$, we define the real vector space $\ll x \gg$ by

$$\ll x \gg := \operatorname{span}_{\mathbb{R}}\{x^{(2p+1)} : p = 0, 1, 2, \dots\}.$$

These two notions allows us to state the following:

Result 5.2.2 (Spectral decomposition theorem)**.** *Let $(V, \{\cdot,\cdot,\cdot\})$ be a finite-dimensional PHJTS. Then, each $x \in V \setminus \{0\}$ can be written uniquely as*

$$x = \lambda_1 e_1 + \cdots + \lambda_s e_s \tag{5.3}$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_s > 0$ and $\{e_1, \dots, e_s\}$ is a $\mathbb{R}$-basis of $\ll x \gg$ comprising pairwise orthogonal tripotents.

The decomposition of $x \in V$ as given by Result 5.2.2 is called the *spectral decomposition* of x. The assignment $x \longmapsto \lambda_1(x)$, where $\lambda_1(x)$ is as given by (5.3), is a well-defined function and can be shown to be a norm on V. This norm is called the *spectral norm* on V.

Next, we present another decomposition, which give us the second ingredient needed to describe the boundary geometry of a bounded symmetric domain.

Result 5.2.3 (Pierce decomposition)**.** *Let $(V, \{\cdot,\cdot,\cdot\})$ be a HJTS and let $e \in V$ be a tripotent. Then, the spectrum of $\mathsf{D}(e,e)$ is a subset of $\{0,1,2\}$. Let*

$$V_j = V_j(e) := \{x \in V : \mathsf{D}(e,e)x = jx\}, \quad j \in \mathbb{Z}.$$

Then:

(i) $V = V_0 \oplus V_1 \oplus V_2$.

(ii) *If $e \neq 0$, then $e \in V_2$.*

(iii) *We have the relation $\{V_\alpha, V_\beta, V_\gamma\} \subset V_{\alpha-\beta+\gamma}$.*

(iv) *V_0, V_1 and V_2 are Hermitian Jordan subsystems of $\{\cdot,\cdot,\cdot\}$.*

The direct-sum decomposition (a) given by the above result is called the *Pierce decomposition of V with respect to the tripotent e*. The ideas that go into proving the Pierce decomposition theorem allow us to construct a special partial order on the set of tripotents of V. In order to avoid statements that are vacuously true, unless stated otherwise, we take $(V, \{\cdot,\cdot,\cdot\})$ to be a PHJTS. Let $e, e' \in V$ be tripotents. We say that *e is dominated by e'* ($e \preceq e'$) if there is a tripotent e_1 orthogonal to e such that $e' = e + e_1$. We say that *e is strongly dominated by e'* ($e \prec e'$) if $e \preceq e'$ and $e \neq e'$. The result of interest, in this regard, is the following:

Result 5.2.4. *Let $(V, \{\cdot,\cdot,\cdot\})$ be a HJTS. Let $e_1, e_2 \in V$ be orthogonal tripotents and let $e = e_1 + e_2$. If $e' \in V$ is a tripotent orthogonal to e, then e' is orthogonal to e_1 and e_2.*

Now suppose $\{\cdot,\cdot,\cdot\}$ is positive. Then, the relation $\preceq$ is a partial order on the set of tripotents.

Definition 5.2.5. A tripotent is said to be *minimal* (or *primitive*) if it is minimal for $\preceq$ among non-zero tripotents. It is said to be *maximal* if it is maximal for $\preceq$.

Result 5.2.6. *Consider the tripotents of V partially ordered by $\preceq$.*

1. *A tripotent e is maximal if and only if the Pierce space $V_0(e) = 0$.*
2. *If, for a tripotent e, the Pierce space $V_2(e) = \mathbb{C}e$, then e is primitive.*

Let us now also assume that $(V, \{\cdot,\cdot,\cdot\})$ is finite dimensional. Given any non-zero tripotent e, it follows from finite-dimensionality and the repeated application of Result 5.2.4 that e can be written as a sum of mutually orthogonal primitive tripotents. This brings us to the final concept in this primer: the *rank of a tripotent e* is the minimum number of primitive tripotents required for such a decomposition of e while the *rank of* $(V, \{\cdot,\cdot,\cdot\})$ is the highest rank that a tripotent of V can have.

5.3 The boundary geometry of bounded symmetric domains

In this section we describe the boundary of a bounded symmetric domain in terms of the positive Hermitian Jordan triple system associated to it. Thus, we shall follow the notation introduced in Section 5.2. Recall that a bounded symmetric domain D has a realization as a bounded convex balanced domain. When we say "Hermitian Jordan triple system associated to D", it is implicit that D is this realization and the association is the one given by (5.1). This is a one-to-one correspondence, described as follows:

Result 5.3.1 ([Loo77], Theorem 4.1)**.** *Let D be a realization of a bounded symmetric domain as a bounded convex balanced domain in $\mathbb{C}^n$ for some $n \in \mathbb{Z}_+$. Then, D is the open unit ball in $\mathbb{C}^n$ with respect to the spectral norm determined by the PHJTS associated to D. Conversely, given a PHJTS $(\mathbb{C}^n, \{\cdot,\cdot,\cdot\})$, the open unit ball with respect to the spectral norm determined by it is a bounded symmetric domain D, and the PHJTS associated to D by the rule* (5.1) *is* $(\mathbb{C}^n, \{\cdot,\cdot,\cdot\})$.

In what follows, whenever we mention a bounded symmetric domain D, *it will be understood that D is a bounded convex balanced realization*.

The boundary of a bounded symmetric domain $D \subset \mathbb{C}^n$ has a certain stratification into real-analytic submanifolds that can be described in terms of the PHJTS associated to D. The first part of this section is devoted to describing this stratification. Fix a bounded symmetric domain $D \subset \mathbb{C}^n$ and let $(\mathbb{C}^n, \{\cdot,\cdot,\cdot\}_D)$ be the PHJTS associated to it. It turns out (see [Loo77, Theorem 5.6]) that the set M_D of tripotents of $\mathbb{C}^n$ with respect to $\{\cdot,\cdot,\cdot\}_D$ is a disjoint union of real-analytic submanifolds of $\mathbb{C}^n$. For each $e \in M_D$, let $M_{D,e}$ denote the connected component of M_D containing e. The tangent space $T_e(M_{D,e})$, viewed *extrinsically* (i.e., so that $e + T_e(M_{D,e})$ is the affine subspace of all tangents to $M_{D,e}$ at e), is:

$$T_e(M_{D,e}) = iA(e) \oplus V_1(e),$$

where $A(e)$ is determined by the relation $V_2(e) = \{x + iy \in \mathbb{C}^n : x, y \in A(e)\}$, and $V_j(e)$ is the eigenspace of $j = 0, 1, 2$ in the Pierce decomposition of $\mathbb{C}^n$ with respect to e.

Let M_D^* be the set of all non-zero tripotents and let $\|\cdot\|_D$ denote the spectral norm determined by $\{\cdot,\cdot,\cdot\}_D$. Define

$$\begin{aligned} E_D &:= \{(e, v) \in \mathbb{C}^n \times \mathbb{C}^n : e \in M_D^* \text{ and } v \in V_0(e)\}, \\ \mathfrak{B}_D &:= \{(e, v) \in E_D : \|v\|_D < 1\}. \end{aligned}$$

We can write $\mathfrak{B}_D$ as a disjoint union of the form

$$\mathfrak{B}_D := \bigsqcup_{\alpha \in C} \mathfrak{B}_{D,\alpha}, \tag{5.4}$$

where C is the set of connected components of M_D^*, and each $\mathfrak{B}_{D,\alpha}$ is a connected, real-analytic submanifold of $\mathbb{C}^n \times \mathbb{C}^n$ that is a real-analytic fibre bundle whose fibres are unit $\|\cdot\|_D$-discs. The key theorem about the boundary of D is as follows:

Result 5.3.2 ([Loo77], Chapter 6)**.** *Let D be a bounded symmetric domain in $\mathbb{C}^n$ and let $\mathsf{f} : \mathfrak{B}_D \to \mathbb{C}^n$ be defined by $\mathsf{f}(e, v) := e + v$. Then:*

- (i) *$\mathsf{f}|_{\mathfrak{B}_{D,\alpha}}$ is an imbedding for each $\alpha \in C$;*
- (ii) *$\partial D = \sqcup_{\alpha \in C} \mathcal{M}_{D,\alpha}$, where $\mathcal{M}_{D,\alpha} := \mathsf{f}(\mathfrak{B}_{D,\alpha})$;*
- (iii) *in the above stratification of ∂D, if $\mathcal{M}_{D,\alpha}$ is of dimension d_α, then it is a closed, connected, real-analytic imbedded submanifold of the open set*

$$\mathbb{C}^n \setminus \bigcup_{\beta : \dim_{\mathbb{R}}(\mathcal{M}_{D,\beta}) < d_\alpha} \mathcal{M}_{D,\beta}.$$

Furthermore, when D is an *irreducible* bounded symmetric domain in $\mathbb{C}^n$, then we can provide further information. Here, the rank of a bounded symmetric domain is the rank of the Jordan triple system $(\mathbb{C}^n, \{\cdot,\cdot,\cdot\}_D)$.

Result 5.3.3 ([Loo77], Chapter 6; [Vig91], Théorème 7.3)**.** *Let D be an irreducible bounded symmetric domain in $\mathbb{C}^n$ of rank r, and let C denote the set of connected components of $\mathfrak{B}_D$. Then, we have the following:*

(i) *C has cardinality r.*

(ii) *Each connected component of the decomposition* (5.4) *is a bundle over a submanifold of non-zero tripotents of rank j, $j \in \{1,\dots,r\}$. Denoting this bundle as $\mathfrak{B}_{D,j}$, $j \in \{1,2,\dots,r\}$, we can express the stratification of ∂D given by Result 5.3.2-(ii) as*

$$\partial D = \bigsqcup_{j=1}^{r} \mathcal{M}_{D,j},$$

where $\mathcal{M}_{D,j} := \mathbf{f}(\mathfrak{B}_{D,j})$, and each $\mathcal{M}_{D,j}$ is connected.

(iii) *The stratum $\mathcal{M}_{D,1}$ is dense in ∂D.*

The other goal of this section is to describe the structure of the germs of complex-analytic varieties contained in the boundary of a bounded symmetric domain D. This structure can be described in extremely minute detail; see, for instance, [Wol72] by Wolf. In fact, the papers about higher-rank bounded symmetric domains mentioned in Section 1.4 make extensive use of this fine structure. However, *in this work, we only need very coarse information about the complex analytic structure of ∂D*; specifically: the distinction between the Shilov boundary of D and its complement in ∂D.

We denote the Shilov boundary of D by $\partial_S D$. Recall that, given a uniform algebra $\mathfrak{A}$ on a compact Hausdorff space X, a *boundary* for $\mathfrak{A}$ is a closed set $S \subset X$ such that

$$\sup_S |f| = \sup_X |f| \quad \forall f \in \mathfrak{A}.$$

It can be proved that the intersection of all the boundaries for $\mathfrak{A}$ is itself a boundary for $\mathfrak{A}$, known as the *Shilov boundary* for $\mathfrak{A}$. In this chapter, whenever we use the term "Shilov boundary", we shall mean the Shilov boundary for the uniform algebra $A(D) := \mathcal{O}(D) \cap \mathcal{C}(\overline{D})$. We will need the following definition which is related to the present discussion:

Definition 5.3.4. Let D be a bounded domain in $\mathbb{C}^n$. An *affine ∂D-component* is an equivalence class under the equivalence relation $\sim_A$ on ∂D given by

$$x \sim_A y \iff x \text{ and } y \text{ can be joined by a chain of segments lying in } \partial D,$$

where a segment is a subset of $\mathbb{C}^n$ of the form $\{u + tv : t \in (0,1)\}$, $u, v \in \mathbb{C}^n$. A *holomorphic arc component of ∂D* is an equivalence class under the equivalence relation $\sim_H$ on ∂D given by

$$x \sim_H y \iff x \text{ and } y \text{ can be joined by a chain of analytic discs lying in } \partial D.$$

Roughly speaking, given a bounded domain $D \Subset \mathbb{C}^n$ and a point $x \in \partial D$, the holomorphic arc component of ∂D containing x is the largest (germ of a) complex-analytic variety lying in ∂D that contains x. The information that we require about holomorphic boundary components is:

Result 5.3.5 ([Loo77], Theorem 6.3)**.** *Let D be the realization of a bounded symmetric domain as a bounded convex balanced domain in $\mathbb{C}^n$.*

(i) *The affine ∂D-components and the holomorphic arc components of ∂D coincide.*

(ii) *A boundary component containing a point $x \in \partial D$ is a non-empty open region in some $\mathbb{C}$-affine subspace of positive dimension passing through x* unless *x is a maximal tripotent.*

Finally, we mention the following description of the Shilov boundary of a bounded symmetric domain:

Result 5.3.6 ([Loo77], Theorem 6.5)**.** *Let $D \Subset \mathbb{C}^n$ be as in Result 5.3.5. The Shilov boundary of D coincides with each of the following sets:*

(i) *the set of maximal tripotents of $\mathbb{C}^n$ with respect to $\{\cdot,\cdot,\cdot\}_D$;*

(ii) *the set of extreme points of $\overline{D}$;*

(iii) *the set of points of $\overline{D}$ having the maximum Euclidean distance from $0 \in \mathbb{C}^n$.*

5.4 Some essential propositions

This section contains several lemmas and propositions — some being simple consequences of known results, and some requiring substantial work — that will be needed to prove our theorems. We begin by giving formulas for certain special automorphisms of a bounded symmetric domain.

Let D be a bounded symmetric domain in $\mathbb{C}^n$. Let $(\mathbb{C}^n, \{\cdot,\cdot,\cdot\}_D)$ be the Jordan triple system associated to D (as in other places in this paper, we assume that D is a Harish-Chandra realization). Let $\mathbf{D}_D$ and Q_D be the maps (5.2) for the triple product $\{\cdot,\cdot,\cdot\}_D$. We define the linear operators $\mathbf{B}_D(x,y) : \mathbb{C}^n \to \mathbb{C}^n$:

$$\mathbf{B}_D(x,y) := \mathsf{id}_D - \mathbf{D}_D(x,y) + Q(x)Q(y), \quad x, y \in \mathbb{C}^n.$$

Consider the sesquilinear form $(x,y) \longmapsto \mathrm{Tr}[\mathbf{D}_D(x,y)]$ on $\mathbb{C}^n$. It turns out that the positivity of $\{\cdot,\cdot,\cdot\}_D$ is equivalent to the above sesquilinear form being an inner product on $\mathbb{C}^n$; see [Loo77, Chapter 3]. Furthermore with respect to this inner product, we have:

$$\mathbf{B}_D(x,y)^* = \mathbf{B}_D(y,x) \ \ \forall x, y \in \mathbb{C}^n.$$

It is now easy to deduce that $\mathbf{B}_D(a,a)$ is a self-adjoint, positive semi-definite linear operator. Consequently, $\mathbf{B}_D(a,a)$ admits a unique positive semi-definite square root, which we denote by $\mathbf{B}_D(a,a)^{1/2}$. Having made these two definitions, we can now give an explicit formula for some special automorphisms of D.

Result 5.4.1 ([Loo77], Proposition 9.8; [Roo00], Proposition III.4.1)**.** *Let D be the realization of a bounded symmetric domain as a convex balanced domain in $\mathbb{C}^n$. Fix a point $a \in D$ and let*

$$g_a(z) := a + \mathbf{B}_D(a,a)^{1/2}(\mathsf{id}_D + \mathbf{D}_D(z,a))^{-1}(z) \ \ \forall z \in D.$$

Then, $g_a \in \mathrm{Aut}(D)$, $g_a(0) = a$, and $g_a'(z) = \mathbf{B}_D(a,a)^{1/2} \circ \mathbf{B}_D(z,-a)^{-1}$. Furthermore, $g_a^{-1} = g_{-a}$.

Various versions of the following lemma have been known for a long time. We refer the reader to [Rud08, Lemma 15.2.2] for a proof.

Lemma 5.4.2. *Let D be a bounded domain in $\mathbb{C}^n$ and let $p \in \partial D$. Assume that there exists a ball B centered at p and a function $h \in \mathcal{O}(B \cap D) \cap \mathcal{C}(\overline{B \cap D}; \mathbb{C})$ such that $h(p) = 1$ and $|h(z)| < 1 \ \forall z \in \overline{B \cap D} \setminus \{p\}$. Let $a_0 \in D$ and $\{\phi_k\}$ be a sequence of automorphisms of D such that $\phi_k(a_0) \longrightarrow p$ as $k \to \infty$. Then, $\{\phi_k\}$ converges uniformly on compact subsets of D to const_p — the map that takes the constant value p.*

We now state a version of Schwarz's lemma for convex balanced domains and then a version of Schwarz's lemma for irreducible bounded symmetric domains, both of which are needed in the proof of our Key Lemma (see Section 5.1).

Result 5.4.3 ([Rud08], Theorem 8.1.2)**.** *Let Ω_1 and Ω_2 be balanced regions in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively, and let $F : \Omega_1 \to \Omega_2$ be a holomorphic map. Suppose Ω_2 is convex and bounded. Then:*

i) *$F'(0)$ maps Ω_1 into Ω_2; and*

ii) *$F(r\Omega_1) \subseteq r\Omega_2$ $(0 < r \leq 1)$ if $F(0) = 0$.*

Result 5.4.4 ([Vig91], Théorème 7.4)**.** *Let D be an irreducible bounded symmetric domain in $\mathbb{C}^n$ in its Harish-Chandra realization (whence it is the unit ball in $\mathbb{C}^n$ for the associated spectral norm $\|\cdot\|$). Let $F : D \to D$ be a holomorphic map such that $F(0) = 0$. Assume that for some non-empty open set $U \subset D$, we have $\|F(z)\| = \|z\| \ \forall z \in U$. Then F is an automorphism of D.*

With these two results, we can now give a proof of our Key Lemma:

The proof of the Key Lemma. Let $z \in U$, and set $w := F(z)$. By hypothesis, F maps Δ_z into D and $(F|_{W_1})^{-1}$ maps Δ_w into D. Applying Result 5.4.3 to $F|_{\Delta_z}$ and to $(F|_{W_1})^{-1}|_{\Delta_w}$, we have $\|F(z)\| = \|z\|$ for every $z \in U$. Thus by the Schwarz lemma for irreducible bounded symmetric domains, F is an automorphism of D. □

We now state and prove a technical proposition regarding the invertibility of the operator $\mathbf{B}_D$ at certain off-diagonal points in $\partial D \times \partial D$, where D is an *irreducible* bounded symmetric domain of dimension ≥ 2. Here $\mathcal{M}_{D,1}$ denotes the stratum of ∂D described by Result 5.3.3. This result and our Key Lemma are the central ingredients in the proof of our Main Theorem.

Proposition 5.4.5. *Let D be the realization of an irreducible bounded symmetric domain of dimension n as a bounded convex balanced domain in $\mathbb{C}^n$, $n \geq 2$. Let $p \in \partial D$. For each $z_0 \in \mathcal{M}_{D,1}$ and each $\mathcal{M}_{D,1}$-open neighbourhood $U \ni z_0$, there exists a point $w \in U$ such that* $\det \mathbf{B}_D(\cdot, p)$ *is non-zero on the set $\{\zeta w : \zeta \in \mathbb{C}, |\zeta| = 1\}$.*

Remark 5.4.6. In the following proof, we argue by assuming that the conclusion above is false. We can instantly arrive at a contradiction at the point (•) in the proof below if we invoke results on the fine structure of ∂D; see [Wol72], for instance. However, we provide an elementary argument beyond (•) to complete the proof in the hope that appropriate analogues of the above may be formulated in other contexts.

Proof. Let us denote $\det \mathbf{B}_D(z,p)$ as $h(z)$, where $z \in \mathbb{C}^n$. Let us assume that the result is false. Then, there exists a point $z_0 \in \mathcal{M}_{D,1}$ and an $\mathcal{M}_{D,1}$-open neighbourhood $U \ni z_0$ such that for each $w \in U$, there exists a $\zeta_w \in \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ with $h(\zeta_w w) = 0$. Let q denote the quotient map $q : \mathbb{C}^n \setminus \{0\} \to \mathbb{CP}^{n-1}$. Also write

$$Z_h := h^{-1}\{0\}, \quad Z := Z_h \cap \mathcal{M}_{D,1}.$$

Our assumption implies that $q(Z)$ contains a non-empty open set $\mathcal{V} \subset \mathbb{CP}^{n-1}$. Let $\mathcal{A} := \{z \in \mathbb{C}^n : 1 - \varepsilon < \|z\| < 1 + \varepsilon\}$, where $\|\cdot\|$ denotes the spectral norm relative to which D is the unit ball, and ε is a *fixed* number in $(0,1)$. As $\mathcal{V} \subset q(\mathcal{A})$, it is easy to see that $\mathcal{V}$ can be covered by finitely many holomorphic coordinate patches $(U_1, \psi_1), \dots, (U_M, \psi_M)$ such that the maps

$$q_j := \psi_j \circ q|_{q^{-1}(U_j) \cap \mathcal{A}} : q^{-1}(U_j) \cap \mathcal{A} \to \mathbb{C}^{n-1}$$

are Lipschitz maps. Since Lipschitz maps cannot increase Hausdorff dimension (see [Rud08, Proposition 14.4.4], for instance) and $\dim_{\mathbb{R}}(\mathcal{V}) = 2n - 2$, the preceding discussion shows that the Hausdorff dimension of Z (and hence the dimension of Z as a real-analytic set) is $2n - 2$. As Z_h is a complex analytic subvariety, its singular locus is of complex dimension $\leq n - 2$. Thus, we can find a point $x_0 \in Z$ that is a regular point of Z_h, and an open ball B around x_0 that is so small that

- $\mathcal{M}_{D,1} \cap B$ is a submanifold of B;
- $B \cap Z_h$ is an $(n-1)$-dimensional complex submanifold of B;
- the dimension of $B \cap Z$ is $2n - 2$.

These three facts imply that $M := B \cap Z_h \subset \mathcal{M}_{D,1}$. We can deduce this by considering a local defining function $\rho_B : B \to \mathbb{R}$ for $\mathcal{M}_{D,1}$ and observing that, by Łojasiewicz's theorem [Łoj59], $\rho_B|_M \equiv 0$. *If $D = \mathbb{B}^n$, we already have a contradiction* and, hence, the proof.

Since $\mathcal{M}_{D,1}$ is a real-analytic submanifold of $\mathbb{C}^n \setminus \bigsqcup_{j \geq 2} \mathcal{M}_{D,j}$, where $\mathcal{M}_{D,j}$ are the strata of ∂D discussed in Section 5.3, we can define the Levi-form of $\mathcal{M}_{D,1}$ — denoted by $\mathfrak{L}(z, V)$, $z \in \mathcal{M}_{D,1}$, $V \in H_z(\mathcal{M}_{D,1})$. A few words about notation: in this proof, we shall work with the tangent bundle of $\mathcal{M}_{D,1}$ defined extrinsically. So, when referring to vectors in $T_z(\mathcal{M}_{D,1})$, we shall view them either as real or as complex vectors, as convenient, such that $z + T_z(\mathcal{M}_{D,1})$ is the hyperplane tangent to $\mathcal{M}_{D,1}$ at $z \in \mathcal{M}_{D,1}$. In this scheme:

$$H_z(\mathcal{M}_{D,1}) := T_z(\mathcal{M}_{D,1}) \cap iT_z(\mathcal{M}_{D,1}).$$

As $\dim_{\mathbb{C}}(M) = n-1$, $\mathfrak{L}(z,\cdot) \equiv 0\ \forall z \in M$. The curve $\gamma : (-\varepsilon,\varepsilon) \to \mathcal{M}_{D,1}$ (for $\varepsilon > 0$ suitably small) $\gamma(t) := \exp(it)z$ is transverse to M at z. This is because if $\gamma'(0) = iz$ were in $H_z(\mathcal{M}_{D,1})$, then

$$i\gamma'(0) = -z \in H_z(\mathcal{M}_{D,1}) \subset T_z(\mathcal{M}_{D,1}),$$

which contradicts the convexity of D. Consequently, for $\varepsilon_0 > 0$ sufficiently small, the set $\{\exp(it)z : t \in (-\varepsilon_0,\varepsilon_0), z \in M\}$ contains an $\mathcal{M}_{D,1}$-open neighbourhood of x_0. Thus, $\mathcal{M}_{D,1}$ is Levi-flat at x_0. As $\mathcal{M}_{D,1}$ is real-analytic, it is a Levi-flat hypersurface.

We shall now show that Levi-flatness of $\mathcal{M}_{D,1}$ leads to a contradiction. Let us pick an $x \in \mathcal{M}_{D,1}$. Owing to Levi-flatness, we can find a ball B_x, centered at x, such that

$$D_x^- := D \cap B_x, \quad D_x^+ := B_x \setminus \overline{D}$$

are both pseudoconvex. Let $\boldsymbol{n}_x$ denote the unit outward normal vector to ∂D at x ($x \in \mathcal{M}_{D,1}$). Owing to convexity of D, we can find an $\varepsilon_0 > 0$ and a $\delta_0 > 0$ such that

$$H_x(\varepsilon_0;\delta) := x + \delta\boldsymbol{n}_x + \{V \in H_x(\mathcal{M}_{D,1}) : |V| < \varepsilon_0\} \subset D_x^+$$

for each $\delta \in (0,\delta_0)$. Here, $|\cdot|$ denotes the Euclidean norm. As $H_x(\varepsilon_0;\delta)$ is a copy of a complex $(n-1)$-dimensional ball and as D_x^+ is taut — see [KR81, Proposition 2.1] — it follows that $H_x(\varepsilon_0;0) \subset \mathcal{M}_{D,1}$. To summarize, $\mathcal{M}_{D,1}$ has the following property:

($\bullet$) At each $x \in \mathcal{M}_{D,1}$, a germ of the set $(x + H_x(\mathcal{M}_{D,1}))$ lies in $\mathcal{M}_{D,1}$.

Let us now pick and fix a point $y^0 \in \mathcal{M}_{D,1}$. Let $(z_1,\dots,z_n)$ be global holomorphic coordinates in $\mathbb{C}^n$, associated to an appropriate rigid motion of D, such that $y^0 = (0,\dots,0), D \subset \{\mathsf{Re} z_1 > 0\}$ and $H_{y^0}(\mathcal{M}_{D,1}) = \{z_1 = 0\}$ relative to these coordinates. Let W be a non-zero vector in $H_{y^0}(\mathcal{M}_{D,1})$ and let $D_W := D \cap \mathrm{span}_{\mathbb{C}}\{W,\boldsymbol{n}_{y^0}\}$. Clearly, D_W is convex and by ($\bullet$) $\mathcal{M}_{D,1} \cap \mathrm{span}_{\mathbb{C}}\{W,\boldsymbol{n}_{y^0}\} =: \mathcal{M}_W$ has the property that for each point $y \in \mathcal{M}_W$, the germ of a complex line through y, call it $\Lambda_{y,W}$, lies in $\mathcal{M}_W$. Let us view D_W as lying in $\mathbb{C}^2$, whence a portion of $\mathcal{M}_W$ near $(0,0)$ can be parametrized by three real variables as follows:

$$r(t,u,v) = \rho(t) + a(t)(u+iv), \quad |t| < \varepsilon_1,\ |u|,\ |v| < \varepsilon_2,$$

where $\rho : (-\varepsilon_1,\varepsilon_1) \to \mathcal{M}_W$ is a smooth curve through $(0,0)$ such that $\rho'(t)$ is orthogonal to $\Lambda_{\rho(t),W}$ for each t, and $a : (-\varepsilon_1,\varepsilon_1) \to \mathbb{C}^2$ is such that $a(t)$ is parallel to $\Lambda_{\rho(t),W}$ for each t. For the remainder of this paragraph, $\boldsymbol{n}(t,u,v)$ will denote the *inward* unit normal to ∂D_W

at $r(t,u,v)$, and $\cdot$ will denote the standard inner product on $\mathbb{R}^4$. Define the matrix-valued function $\Gamma : (-\varepsilon_1,\varepsilon_1) \times (-\varepsilon_2,\varepsilon_2)^2 \to \mathbb{R}^{3\times 3}$ by

$$\Gamma(\tau,U,V) := \mathsf{Hess}_{t,u,v}\,(r(t,u,v)\cdot \boldsymbol{n}(\tau,U,V))\big|_{(t,u,v)=(\tau,U,V)}\ .$$

The convexity of D_W implies that $\Gamma(\tau,U,V)$ is *positive* semidefinite at each (τ,U,V) (recall that $\boldsymbol{n}(\tau,U,V)$ is the inward normal at $r(\tau,U,V)$). By choosing $\varepsilon_1,\varepsilon_2 > 0$ small enough, we can ensure that $(n_1^2+n_2^2)(t,u,v) \neq 0$ for every (t,u,v), where we write $\boldsymbol{n} = (n_1,n_2,n_3,n_4)$, and that a is of the form $a(t) = (\alpha(t)+i\beta(t),1)$. We compute to observe that two of the principal minors of Γ turn out to be $-(n_1\alpha'+n_2\beta')^2$ and $-(n_2\alpha'-n_1\beta')^2$, which must be non-negative. This gives us the system of equations

$$\begin{aligned} n_1\alpha' + n_2\beta'|_{(\tau,U,V)} &= 0\\ -n_1\beta' + n_2\alpha'|_{(\tau,U,V)} &= 0 \ \ \forall(\tau,U,V). \end{aligned}$$

By our assumption on $\boldsymbol{n}$, this implies that $\alpha' = \beta' \equiv 0$. Restating this geometrically, there is a small $\mathcal{M}_W$-open neighbourhood of $0 \in \partial D_W$ such that, for every y in this neighbourhood, $\Lambda_{y,W}$ is parallel to the vector W. This holds true for each non-zero $W \in H_{y^0}(\mathcal{M}_{D,1})$. Thus, there is an $\mathcal{M}_{D,1}$-open patch $\omega \ni y^0$ such that

$$x + H_x(\mathcal{M}_{D,1}) \text{ is parallel to } \{z_1 = 0\} \text{ for every } x \in \omega. \tag{5.5}$$

By Result 5.3.3, $\mathcal{M}_{D,1}$ is connected. Thus, if $y^0 \neq y \in \mathcal{M}_{D,1}$, then y can be joined to y^0 by a chain of $\mathcal{M}_{D,1}$-open patches $\Omega_0,\dots,\Omega_N$, where Ω_0 equals the patch Ω in (5.5), $\Omega_{j-1}j\cap\Omega_j \neq \emptyset$, $j = 1,\dots,N$, and $\Omega_N \ni y$. By a standard argument of real-analytic continuation, we deduce that (5.5) holds with Ω_N replacing Ω (where z_1 comes from the global system of coordinates fixed at the beginning of the previous paragraph). Hence, $x + H_x(\mathcal{M}_{D,1})$ is parallel to $\{z_1 = 0\}$ for each $x \in \mathcal{M}_{D,1}$. As $\mathcal{M}_{D,1}$ is dense in ∂D, and D is bounded, we can find a $\xi \in D$ and a vector $W = (W_1,\dots,W_n)$ with $W_1 = 0$ such that the ray $\{\xi + tW; t \geq 0\}$ intersects ∂D at a point in $\mathcal{M}_{D,1}$. Then, this ray must be tangential to $\mathcal{M}_{D,1}$ at the point of intersection, which is absurd as D is convex. Hence, our initial assumption must be false. $\square$

5.5 The proof of Theorem 5.1.1

Before we proceed further, we clarify our notation for the different norms that will be used in the proof of Theorem 5.1.1. With D_1 and D_2 as in Theorem 5.1.1, $\|\cdot\|_j$ will denote the spectral norms such that D_j is the unit $\|\cdot\|_j$-ball in $\mathbb{C}^n, j = 1,2$. The Euclidean norm on $\mathbb{C}^n$ will be denoted by $|\cdot|$. We will also need to impose norms on certain linear operators on $\mathbb{C}^n$. We shall use the operator norm induced by the Euclidean norm: for a $\mathbb{C}$-linear operator A on $\mathbb{C}^n$, we set

$$\|A\|_{op} := \sup_{|x|=1} |Ax|.$$

The proof of Theorem 5.1.1. We shall take D_1 and D_2 to be Harish-Chandra realizations of the given bounded symmetric domains. We may assume, composing F with suitable automorphisms if necessary, that $F(0) = 0$.

By Lemma 2.2.5, F extends to a holomorphic map defined on a neighbourhood N of $\overline{D_1}$. For simplicity of notation, we shall denote this extension also as F. The complex Jacobian $\mathsf{Jac}_{\mathbb{C}}F$ is holomorphic on N and $\mathsf{Jac}_{\mathbb{C}}F \not\equiv 0$ on D_1. Hence, by the maximum principle, $\mathsf{Jac}_{\mathbb{C}}F \not\equiv 0$ on ∂D_1. By definition, we can find a point $p \in \partial_S D_1$ such that

$$\sup_{\overline{D_1}} |\mathsf{Jac}_{\mathbb{C}}F| = |\mathsf{Jac}_{\mathbb{C}}F(p)| \neq 0.$$

By the inverse function theorem, we can find a ball $B(p,r) \subset N$ such that $F|_{B(p,r)}$ is injective. Let us write

$$\Omega_1 := B(p,r) \cap D_1, \ \ \Omega_2 := F(B(p,r)) \cap D_2.$$

We shall use our Key Lemma (see Section 5.1, and Section 5.4 for its proof) to deduce the result. The regions W_1 and W_2 of that Lemma will be constructed by applying suitable automorphisms to Ω_1 and Ω_2.

Claim. $F(p) \in \partial_S D_2$.

Suppose $F(p) \notin \partial_S D_2$. It follows from Result 5.3.5 and Result 5.3.6 that there is a vector $V \in \mathbb{C}^n \setminus \{0\}$ and neighbourhood Ω of $0 \in \mathbb{C}$ such that $\psi(\Omega) \subset F(B(p,r)) \cap \partial D_2$, where $\psi : \Omega \ni \zeta \longmapsto F(p) + \zeta V$. Next, define

$$\widetilde{\psi} := (F|_{B(p,r)})^{-1} \circ \psi.$$

Since $F|_{D_1}$ is proper and $F|_{B(p,r)}$ is injective,

$$F(z) \in F(B(p,r)) \cap \partial D_2 \iff z \in B(p,r) \cap \partial D_1.$$

Thus $\widetilde{\psi}(\Omega) \subset \partial D_1$. Furthermore, $\widetilde{\psi}$ is non-constant and $\widetilde{\psi}(0) = p$. By definition, each point of $\widetilde{\psi}(\Omega) \setminus \{p\}$ lies in the holomorphic arc component of ∂D_1 containing p. This is a contradiction since p, being an extreme point, is a one-point affine ∂D_1-component and thus, by Result 5.3.5, a one-point holomorphic arc component of ∂D_1. Hence the claim.

Let us now take a sequence $\{a_k\} \subset \Omega_1$ such that $a_k \to p$, and let $b_k := F(a_k)$. Let $\phi_k^1 \in \mathrm{Aut}(D_1)$ denote an automorphism that maps 0 to a_k. Let $\phi_k^2 \in \mathrm{Aut}(D_2)$ be an automorphism that maps 0 to b_k. Owing to Result 5.3.6 and to convexity, we can construct a peak function for p on $\overline{D_1}$. Likewise (in view of the last claim) $F(p)$ is a peak point of D_2. By Lemma 5.4.2, we get:

$$\phi_k^j \longrightarrow \mathsf{const}_{p^j} \text{ uniformly on compacts}, \ \ j = 1,2, \tag{5.6}$$

where $p^j, p^1 := p, p^2 := F(p)$.

We now define

$$\Omega_j^k := (\phi_k^j)^{-1}(\Omega_j), \ \ j = 1,2, \ k \in \mathbb{Z}_+.$$

Given any $r > 0$, write $rD_j := \{z \in \mathbb{C}^n : \|z\|_j < r\}$, $j = 1,2$. By (5.6), there exists a sequence $k_1 < k_2 < k_3 < \dots$ in $\mathbb{Z}_+$ such that

$$\phi^1_{k_\nu}\left((1-1/s)\overline{D}_1\right) \subset \Omega_1 \;\; \forall \nu \geq s,\; s \in \mathbb{Z}_+.$$

By (5.6) again, we can extract a sequence of indices $\nu(1) < \nu(2) < \nu(3) < \dots$ such that

$$\phi^2_{k_{\nu(t)}}\left((1-1/s)\overline{D}_2\right) \subset \Omega_2 \;\; \forall t \geq s,\; s \in \mathbb{Z}_+.$$

In the interests of readability of notation, let us re-index $\{k_{\nu(s)}\}_{s\in\mathbb{Z}_+}$ as $\{k_m\}_{m\in\mathbb{Z}_+}$. Then, the above can be summarized as:

(∗) With the sequences of maps $\{\phi^1_k\} \subset \mathrm{Aut}(D_1)$ and $\{\phi^2_k\} \subset \mathrm{Aut}(D_2)$ as described above, there is a sequence $\{k_m\}_{m\in\mathbb{Z}_+} \subset \mathbb{Z}_+$ and a strictly increasing $\mathbb{Z}_+$-valued function ν^* such that

$$(1-1/s)\overline{D}_1 \subset \Omega_1^{k_m} \;\; \forall m \geq s,\; s \in \mathbb{Z}_+, \tag{5.7}$$

$$(1-1/\nu^*(s))\overline{D}_2 \subset \Omega_2^{k_m} \;\; \forall m \geq s,\; s \in \mathbb{Z}_+. \tag{5.8}$$

Step 1. *Analyzing the family* $\{(\phi^2_{k_m})^{-1} \circ F \circ \phi^1_{k_m}\}_{m\in\mathbb{Z}_+}$
Consider the maps $G_m : D_1 \to D_2$ defined by

$$G_m := (\phi^2_{k_m})^{-1} \circ F \circ \phi^1_{k_m}.$$

By Montel's theorem, and passing to a subsequence and relabelling if necessary, we get a map $G \in \mathcal{O}(D_1; \mathbb{C}^n)$ such that $G_m \to G$ uniformly on compact subsets. Let us fix an $s \in \mathbb{Z}_+$. By (∗), we infer that $\exists M_s \in \mathbb{Z}_+$ such that $(1-1/s)\overline{D}_j \subset \Omega_j^{k_m}$ $\forall m \geq M_s$, $j = 1,2$. Note that $G_m|_{\Omega_1^{k_m}}$ is a biholomorphism, whence $G'_m(0)$ is invertible for each m. Hence, by the Schwarz lemma for convex balanced domains (i.e. Result 5.4.3 above) $G'_m(0)$ maps $(1-1/s)D_1$ into D_2 and $G'_m(0)^{-1}$ maps $(1-1/s)D_2$ into D_1 $\forall m \geq M_s$. We claim that this implies that $G'(0)$ is invertible. Suppose not. Then we would find a z_0 with $\|z_0\|_1 = (1-2/s)$ such that $G'(0)z_0 = 0$. Note that $G'_m(0) \to G'(0)$ in norm, whence, given any $\varepsilon > 0, \|G'_m(0)z_0\|_2 < \varepsilon$ for every sufficiently large m. If we now choose $\varepsilon \leq (1-2/s)^2$, we see that

$$G'_m(0)^{-1}\left(\{\|w\|_2 = (1-2/s)\}\right) \not\subset D_1$$

for all sufficiently large m. This is a contradiction. Hence the claim.

Now that it is established that $G'(0)$ is invertible, it follows that $G'_m(0)^{-1} \to G'(0)^{-1}$ in norm. Hence, $G'(0)^{-1}$ maps $(1-1/s)D_2$ into D_1. Recall that $s \in \mathbb{Z}_+$ was arbitrarily chosen and that the function ν^* in (∗) is strictly increasing. Thus, $G'(0)^{-1}$ maps D_2 into D_1. By construction, $G(D_1) \subset \overline{D}_2$. Now, D_2 is complete (Kobayashi) hyperbolic. Hence

D_2 is taut; see [Kie70]. As $G(0) = 0 \in D_2, G$ maps D_1 to D_2. So, the holomorphic map $G'(0)^{-1} \circ G : D_1 \to D_1$ satisfies all the conditions of Cartan's uniqueness theorem. Thus,

$$G'(0)^{-1} \circ G = \mathsf{id}_{D_1},$$

which means that $G = G'(0)|_{D_1}$.

Step 2. *Showing that D_1 and D_2 are biholomorphically equivalent*
We have shown in Step 1 that $G'(0)^{-1}$ maps $(1 - 1/s)D_2$ into D_1. As $G'(0)$ is injective, this means that $G'(0)(D_1)$ contains $(1 - 1/s)D_2$ for arbitrarily large $s \in \mathbb{Z}_+$. Thus G maps D_1 onto D_2. It follows that D_1 is biholomorphic to D_2.

It would help to simplify our notation somewhat. By the nature of the argument in Step 1, it is clear that we can assume that the sequences $\{a_k\} \subset \Omega_1$ and $\{b_k\} \subset \Omega_2$ are so selected that $(*)$ is true with $\{k_m\}_{m\in\mathbb{Z}_+} = \{1,2,3,\dots\}$. Owing to Step 2, we may now assume $D_1 = D_2 := D$. The argument we will make in Step 3 below is valid regardless of the specific sequence $\{a_k\}$ or $\{b_k\}$. Hence, in the next three paragraphs following this, the sequence $\{A_k\}$ will stand for either $\{a_k\}$ or $\{b_k\}$, and the point q will stand for either p or $F(p)$. Also, we will abbreviate $\phi^j_{A_k}$ to ϕ_k.

Step 3. *Producing subsequences of $\{\phi_k\}$ that converge on "large" subsets of ∂D.*
By Result 5.4.1 we may take $\phi_k = g_{A_k}$, whence

$$\phi'_k(z) = \mathbf{B}_D(A_k, A_k)^{1/2} \circ \mathbf{B}_D(-z, A_k)^{-1}. \tag{5.9}$$

In the argument that follows, it is implicit that each ϕ_k is defined as a holomorphic map on some neighbourhood (which depends on ϕ_k) of $\overline{D}$; see Lemma 2.2.5. By Proposition 5.4.5 we can find a point $\xi_0 \in \mathcal{M}_{D,1}$ such that

$$\det \mathbf{B}_D(e^{i\theta}\xi_0, q) \neq 0 \ \ \forall\theta \in \mathbb{R}.$$

By continuity, there exists a $\overline{D}$-open neighbourhood Γ of q, an $\mathcal{M}_{D,1}$-open neighbourhood W of ξ_0, and a $\overline{D}$-open set V with the following properties:

(a) $z \in V \implies e^{i\theta}z \in V \ \forall\theta \in \mathbb{R}$;

(b) $V \cap \partial D = S^1 \cdot W$;

(c) $z \in V \implies tz \in V \ \forall t \in [1, 1/\|z\|]$

(now $\|\cdot\|$ is the spectral norm associated to D); such that

$$\det \mathbf{B}_D(z, w) \neq 0 \ \ \forall (z, w) \in \overline{V} \times \Gamma. \tag{5.10}$$

Here, given a set $X \subset \mathbb{C}^n, S^1 \cdot X$ stands for the set $\{e^{i\theta}x : x \in X, \theta \in \mathbb{R}\}$. Let us call any pair (V, W), where V is a $\overline{D}$-open set and W is an $\mathcal{M}_{D,1}$-open set, a *truncated prism with base* $S^1 \cdot W$ if (V, W) satisfies properties (a)-(c) above.

We can find V' and W', with $\overline{W'} \subset W$, such that (V', W') is a truncated prism with base $S^1 \cdot W'$ with the properties:

- $\overline{V'} \subset V$;
- There exists a $\delta_0 \ll 1$ such that for $z_1, z_2 \in \overline{V'}$, the segment $[z_1, z_2] \subset V$ whenever $|z_1 - z_2| < \delta_0$.

Owing to holomorphicity and convexity,

$$\phi_k(z_1) - \phi_k(z_2) = \int_0^1 \phi_k' \left(z_1 + t(z_2 - z_1)\right)(z_2 - z_1)dt, \quad z_1, z_2 \in \overline{D}. \tag{5.11}$$

We can find a $K \equiv K(W)$ such that, in view of (5.10), $\{\mathbf{B}_D(z, A_k) : k \geq K(W), z \in \overline{V}\}$ is a compact family in $GL(n, \mathbb{C})$. Hence, in view of (5.9) (and since $\{\mathbf{B}_D(A_k, A_k) : k \in \mathbb{Z}_+\}$ is a relatively compact family in $\mathbb{C}^{n\times n}$), there exists a constant $C > 0$ such that

$$\|\phi_k'(z)\|_{op} \leq C \; \forall z \in \overline{V}, \;\; \forall k \geq K.$$

By our construction of V', and from (5.11), we conclude:

$$|\phi_k(z_1) - \phi_k(z_2)| \leq C|z_1 - z_2| \;\; \forall z_1, z_2 \in \overline{V'},\ |z_1 - z_2| < \delta_0, \text{ and } \forall k \geq K.$$

In short, $\{\phi_k|_{\overline{V'}}\} \subset C(\overline{V'}; \mathbb{C}^n)$ is an equicontinuous family.

By the Arzela-Ascoli theorem, we can find a subsequence of $\{\phi_k\}$ that converges uniformly to q on $\overline{V'}$. For simplicity of notation, let us continue to denote this subsequence as $\{\phi_k\}$. Then there exists a $K_1 \in \mathbb{Z}_+$ such that $\phi_k(\overline{V'}) \subset \Omega$ (which denotes either Ω_1 or Ω_2) $\forall k \geq K_1$. Furthermore, we may assume that K_1 is so large that, thanks to $(*)$,

$$(1 - 1/s)\overline{D} \subset \phi_k^{-1}(\Omega) \;\; \forall k \geq K_1,$$

where s is so large that $(1 - 1/s)\overline{D} \cap V'$ is a non-empty open set. By construction:

$$z \in V' \cap D \implies \Delta_z \subset (1 - 1/s)\overline{D} \cup V'.$$

Hence $\Delta_z \subset \phi_k^{-1}(\Omega)$ $\forall k \geq K_1$. We summarize the content of this paragraph as follows:

$(**)$ Given any truncated prism (V, W) with base $S^1 \cdot W$ such that $\mathbf{B}_D(z, A_k) \neq 0$ on $\overline{V}$ for all k sufficiently large, we can find a $K_1 \in \mathbb{Z}_+$ and a truncated prism (V', W') with $\overline{V'} \subset V$ such that $\Delta_z \subset \phi_k^{-1}(\Omega)$ for each $z \in V' \cap D$ and each $k \geq K_1$.

Step 4. *Completing the proof.*
By Proposition 5.4.5 and $(**)$, we can find a truncated prism (V', W') with base $S^1 \cdot W'$ which has all the properties stated in $(**)$. Let $s \in \mathbb{Z}_+$ be so large that $(1 - 1/s)D \cap V' := U'$ is a non-empty open set. As $G_k \to G$ uniformly on U' (by Step 1), there exists a point $w_0 \in G(U')$, $K_2 \in \mathbb{Z}_+$ and a $c > 0$ such that the ball

$$B(w_0, c) \subset G(U') \cap G_k(U') \;\text{ and }\; B(w_0, c) \subset \Omega_2^k \;\; \forall k \geq K_2.$$

Write $\|\cdot\|$ for the spectral norm associated to D. Let $R : \mathbb{C}^n \setminus 0 \to \partial D$ be given by $R(w) := w/\|w\|$. By Proposition 5.4.5 and (∗∗), we can find a $\mathcal{M}_{D,1}$-open subset Ω_2 such that

$$\Omega_2 \subset R(B(w_0,c)),$$

a truncated prism (V_2,Ω_2) with base $S^1\cdot\Omega_2$, and a $K_3 \in \mathbb{Z}_+$ such that $\Delta_w \subset \Omega_2^k$ for each $w \in V_2 \cap D$ and each $k \geq K_3$. Let us now set $U := G^{-1}\left(R^{-1}(\Omega_2)\cap B(w_0,c)\right)$, and $K^* := \max(K_1,K_2,K_3)$. Finally, we set

$$W_j := \left(\phi^j_{K^*}\right)^{-1}(\Omega_j), \quad j = 1,2,$$

with the understanding that $\phi^1_k = g_{a_k}$ and $\phi^2_k = g_{b_k}$.

As $U \subset V', \Delta_z \subset W_1$ for each $z \in U$. By construction

$$G_{K^*}(z) \in B(w_0,c) \subset W_2 \ \ \forall z \in U.$$

Finally, by construction, for each $z \in U$, there exists a point $w_z \in \Delta_{G_{K^*}(z)}$ that belongs to $V_2 \cap D$. Thus, $\Delta_{G_{K^*}(z)} \subset W_2$. Recall that $G_{K^*}|_{W_1} : W_1 \to W_2$ is a biholomorphism and $G_{K^*}(0) = 0$. By our Key Lemma, G_{K^*}, and consequently F, must be a biholomorphism. □

5.6 The proof of Theorem 5.1.3

As p is an orbit accumulation point, there is a point $a_0 \in D_1$ and a sequence $\{\phi_k\} \subset \mathrm{Aut}(D_1)$ such that $\phi_k(a_0) \to p$. Regardless of whether p is a peak point or $F(p)$ is a peak point, let us denote the relevant peak function as H. Let B denote a small ball centered at p, with $B \Subset U$, if p is a peak point, and centered at $F(p)$, with $B \Subset F(U)$, if $F(p)$ is a peak point. Depending on whether p or $F(p)$ is a peak point, set $G := F^{-1}$ or $G := F$, respectively. Finally, set

$$h := \begin{cases} H \circ G|_{\overline{B\cap D_2}}, & \text{if } p \text{ is a peak point,} \\ H \circ G|_{\overline{B\cap D_1}}, & \text{if } F(p) \text{ is a peak point.} \end{cases}$$

By our hypothesis on F, it follows that h satisfies all the conditions required of the function h in Lemma 5.4.2 for the appropriate choice of (D,p) depending on whether p or $F(p)$ is a peak point.

Let us now denote the automorphisms discussed above as ϕ^1_k, $k = 1,2,3,\ldots$ Then, using H or the function h constructed above, depending on whether p or $F(p)$ is a peak point, we deduce by Lemma 5.4.2 that $\phi^1_k \longrightarrow \mathsf{const}_p$ uniformly on compact subsets of D_1. Set $a_k := \phi^1_k(0)$. As $a_k \to p$, we may assume without loss of generality that $a_k \in U$. Let $b_k := F(a_k)$, and let $\phi^2_k \in \mathrm{Aut}(D_2)$ be an automorphism that maps 0 to b_k (which is possible as $\mathrm{Aut}(D_2)$ acts transitively on D_2). Repeating the above argument, $\phi^2_k \longrightarrow \mathsf{const}_{F(p)}$ uniformly on compact subsets of D_2. We have arrived at the same result as in (5.6). Thereafter, if we define

$$\Omega_j^k := (\phi_k^j)^{-1}(\Omega_j), \quad j = 1,2, \ k \in \mathbb{Z}_+,$$

where $\Omega_1 := U$ and $\Omega_2 := F(U)$, then, reasoning exactly as in the passage following (5.6), we deduce that $(*)$ from Section 5.5 holds true for our present set-up.

With $\{k_m\}_{m\in\mathbb{Z}_+}$ as given by $(*)$, let us define the maps $G_m : \Omega_1^{k_m} \to \Omega_2^{k_m}$ by

$$G_m := (\phi_{k_m}^2)^{-1} \circ F \circ \phi_{k_m}^1.$$

By construction, each G_m is a biholomorphic map. In particular,

$$G_m(0) = 0, \text{ and } G_m'(0) \in GL(n,\mathbb{C}). \tag{5.12}$$

We may assume, owing to (5.6), that *the sequences* $\{\Omega_j^{k_m}\}_{m\in\mathbb{Z}_+}$ *are increasing sequences*. By Montel's theorem, and arguing by induction, we can find sequences $\{G_{l,m}\}$ and holomorphic maps $\Gamma_l : \Omega_1^{k_l} \to \overline{D}_2$ such that:

- $\{G_{1,m}\}_{m\in\mathbb{Z}_+}$ is a subsequence of $\{G_\nu\}_{\nu\in\mathbb{Z}_+}$ and $\{G_{l+1,m}\}_{m\in\mathbb{Z}_+}$ is a subsequence of $\{G_{l,\nu}\}_{\nu\in\mathbb{Z}_+}$;
- $G_{l,m}|_{\Omega_1^{k_l}} \longrightarrow \Gamma_l$, as $m \to \infty$, uniformly on compact subsets of $\Omega_1^{k_l}$;

for $l = 1,2,3,\dots$ Owing to this construction, the rule

$$\Gamma(z) := \Gamma_l(z) \text{ if } z \in \Omega_1^{k_l},$$

gives a well-defined holomorphic map $\Gamma : D_1 \to \overline{D}_2$.

Let us define $H_l := G_{l,l}$. Now suppose $\Gamma(D_1) \cap \partial D_2 \neq \emptyset$. Then, $\exists \xi \in D_1$ such that $\Gamma(\xi) \in \partial D_2$. Let $M \in \mathbb{Z}_+$ be so large that $\Omega_1^{k_M} \ni \xi$. As D_2 is a bounded symmetric domain, it is taut. Thus, by focusing attention on the sequence

$$\{H_l|_{\Omega_1^{k_M}} : l = M, M+1, M+2, \dots\} \subset \mathcal{O}(\Omega_1^{k_M}; D_2),$$

we must conclude, by assumption, that $\Gamma(\Omega_1^{k_M}) \subset \partial D_2$. But, by (5.12), $\Gamma(0) = 0 \notin \partial D_2$. This is a contradiction, from which we infer:

(a) The range of Γ is a subset of D_2.

Now observe that, by $(*)$, we have:

(b) The sequence $\{H_l : l = s, s+1, s+2, \dots\}$ converges uniformly to Γ on $(1 - 1/s)\overline{D}_1$, $s \in \mathbb{Z}_+$.

(c) H_l^{-1} maps 0 to 0 and $(1 - 1/\nu^*(l))D_2$ into D_1 (since $\mathsf{dom}(H_l^{-1}) = \mathsf{range}(H_l) \supseteq \Omega_2^{k_l}$).

In view of (5.12) and the fact that D_1 and D_2 are balanced, (a)-(c) are precisely the ingredients required to to repeat the argument in Step 1 of the proof of Theorem 5.1.1 to infer that $\Gamma'(0)$ is invertible, $\Gamma'(0)^{-1} : D_2 \to D_1$ and

$$\Gamma'(0)^{-1} \circ \Gamma = \mathsf{id}_{D_1}.$$

Thus, by (a), $\Gamma'(0)(D_1) \subset D_2$. One of the consequences of repeating the argument contained in Step 1 in Section 5.1.1 is, in view of (c), that $\Gamma'(0)^{-1}$ maps $(1 - 1/\nu^*(l))D_2$ into D_2 for every $l \in \mathbb{Z}_+$. As ν^* is strictly increasing and $\mathbb{Z}_+$-valued, and as $\Gamma'(0)$ is injective, this means that $\Gamma'(0)(D_1)$ contains $(1 - 1/s)D_2$ for *arbitrarily large* $s \in \mathbb{Z}_+$, whence $\Gamma'(0)$ maps D_1 onto D_2. Hence, $\Gamma'(0)|_{D_1}$ is a biholomorphism of D_1 onto D_2. $\square$

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