Some Results on Holomorphic Mappings of Domains in $\mathbb{C}^n$

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ABSTRACT. One of the earliest results in the study of automorphisms and holomorphic mappings of domains was Poincaré's proof of the biholomorphic inequivalence of the ball and the polydisk in $\mathbb{C}^n$, $n \geq 2$. This prompted a detailed study of holomorphic mappings of bounded domains in $\mathbb{C}^n$. We survey a number of these results – many of them due to H. Cartan – in Chapter 1. Extending Poincaré's observation, H. Remmert and K. Stein proved a result that implies that there cannot even be a proper holomorphic map from the polydisc to the ball. In Chapter 2, we focus on the *method* of their proof, and prove two Remmert-Stein-type results. In one of these, we extend the classical Remmert-Stein theorem to a broad class of codomains (which includes the unit ball originally considered by Remmert and Stein). We then focus on a closer study of the unit ball. We give a proof of Alexander's result that every proper holomorphic mapping from the ball in $\mathbb{C}^n$, $n \geq 2$, must be an automorphism. To this end, we focus on the structure of general proper holomorphic mappings, and on their effect on analytic varieties. We conclude with a proof of Alexander's theorem.

Declaration

I hereby declare that the work in this thesis has been carried out by me, in the Integrated Ph.D. Program, under the supervision of Prof. Gautam Bharali, and in the partial fulfillment of the requirements of the Master of Science Degree at the Indian Institute of Science, Bangalore. I further declare that this work has not been the basis for the award of any degree, diploma or any other title elsewhere.

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Contents

CHAPTER 1

Introduction and Some Classical Theorems

In this chapter, we present some results on the complex automorphisms of a bounded domain in $\mathbb{C}^n$. Our goal in the first section is to establish the famous result of Poincaré, which asserts that the ball and polydisk are biholomorphically inequivalent in dimensions higher than 1. Therefore, the direct generalization of the Riemann mapping theorem, does not hold in higher dimensions. We also explicitly compute the automorphism groups of the ball and the polydisk. We then state a theorem of Remmert-Stein which shows that the situation when $n > 1$ is really complicated. The theorem says that if a domain Ω has a "local product structure near a point on the boundary", then there cannot even be a proper map from Ω into the ball.

In the next section, we prove a theorem due to Henri Cartan on normally convergent sequences of automorphisms. The theorem yields a number of results on the structure of the automorphism group of a bounded domain, which have far reaching consequences. We end the chapter by stating a deep theorem, again due to H. Cartan, which says that the automorphism group, given the compact-open topology, becomes a Lie group. Most of the results in this chapter are due to Henri Cartan [**3**].

We now recall some elementary definitions. A complex valued function f defined on an open set $\Omega \subseteq \mathbb{C}^n$ is said to be *holomorphic* if at each point $a \in \Omega$, there corresponds a neighbourhood U of a in which f admits a convergent power series expansion in the variables $z_1, \ldots, z_n$. We say $F = (f_1, \ldots, f_m) : \Omega_1 \to \Omega_2$, Ω_1 and Ω_2 regions in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively, is a *holomorphic map* if each f_j is holomorphic, $1 \leq j \leq m$. The terms *biholomorphism* and *automorphism* are defined in the usual manner .

1.1. Biholomorphic inequivalence of the ball and polydisk

To compute the automorphism groups of the ball and the polydisk, as in the univariate case, we require a version of Schwarz's lemma. The following proposition is an analogue of the infinitesimal version of Schwarz's lemma.

PROPOSITION 1.1 (Cartan [**2**]). *Let D be a bounded domain in $\mathbb{C}^n$ and $F : D \to D$ be a holomorphic map. Suppose for some $p \in D$, we have $F(p) = p$ and $F'(p) = I$. Then $F(z) = z$, $\forall z \in D$.*

PROOF. We may assume, without loss of generality, that $p = 0$. Let $r_1 > 0,\ r_2 < \infty$, be such that $r_1\mathbb{B} \subseteq D \subseteq r_2\mathbb{B}$. The power series expansion of F about 0 is of the form

$$F(z) = z + P_2(z) + \dots \ .$$

where P_k is a n-tuple of homogeneous polynomials of degree k.

Let F^k be the k-th iterate of F; specifically, $F^1 := F,\ F^k := F^{k-1} \circ F$. Suppose $F(z) \not\equiv z$. Let N be the smallest integer such that

$$F(z) = z + P_N(z) + \dots \ ,$$

where $P_N \not\equiv 0$.

By induction on k, we see that

$$F^k(z) = z + kP_N(z) \dots \ .$$

Note that F^k is a map of D into itself. Therefore, by Cauchy's inequalities, the coefficients of each component function of $kP_N(z)$ are less than or equal to $r_2 r_1^{-N}$ in absolute value. As k is arbitrary, $P_N \equiv 0$, a contradiction.

□

DEFINITION 1.2. A set $E \subseteq \mathbb{C}^n$ is said to be *circular* if $e^{i\theta} z \in E$ whenever $z \in E$ and θ is real.

PROPOSITION 1.3 (Cartan [**2**]). *Suppose:*

(i) *Ω_1 and Ω_2 are circular domains,* $0 \in \Omega_1 \cap \Omega_2$*,*
(ii) *F is a biholomorphic map of Ω_1 onto Ω_2,with $F(0) = 0$,*
(iii) *Ω_1 is bounded.*

Then $F(z) = F'(0)z$, i.e. F is a linear transformation. In particular, Ω_2 is bounded.

PROOF. Let $G = F^{-1}$. Fix $\theta \in \mathbb{R}$, and define

$$H(z) = G(e^{-i\theta} F(e^{i\theta} z)) \ \ (z \in \Omega_1)$$

As Ω_1 and Ω_2 are circular, $H(z)$ is a well-defined holomorphic map of Ω_1 into Ω_2 that satisfies $H(0) = 0$ and $H'(0) = I$. By the previous Proposition, $H(z) \equiv z$. Therefore,

$$e^{i\theta} F(H(z)) = F(e^{i\theta} z) = e^{i\theta} F(z),$$

for all $z \in \Omega_1$, and for every real θ. Hence, the linear term in the homogeneous expansion of F is the only one different from 0 and we are done. □

With the above results at our disposal, we can now compute the automorphism group of the unit ball and the unit polydisk. We begin with the ball. Every α in the unit disk of $\mathbb{C}$ corresponds to an automorphism ϕ_a of the unit disk that interchanges α and 0, namely $\phi_\alpha(\lambda) = \frac{(\alpha-\lambda)}{1-\bar{\alpha}\lambda}$. The same thing can be done for the unit ball in $\mathbb{C}^n$. Fix $a \in \mathbb{B}$. Let P_a be the orthogonal projection of $\mathbb{C}^n$ onto the subspace $[a]$ generated by a, and let $Q_a := I - P_a$ be the projection onto the orthogonal complement of $[a]$. For instance, $P_0 = 0$ and for $a \neq 0$

$$P_a z = \frac{\langle z, a\rangle}{\langle a, a\rangle} a, \tag{1.1}$$

where we define $\langle z, a\rangle := \sum_{j=1}^n z_j \overline{a}_j$. Define $s_a := (1 - |a|^2)^{1/2}$ and define

$$\phi_a(z) := \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a\rangle}. \tag{1.2}$$

If $\Omega := \{z \in \mathbb{C}^n | \langle z, a\rangle \neq 1\}$ then $\phi_a : \Omega \to \mathbb{C}^n$ is holomorphic and clearly $\overline{\mathbb{B}} \subseteq \Omega$ as $|a| < 1$. Note that when $n = 1$, (1.2) reduces to an automorphism of the unit disk. The following theorem summarizes the properties of ϕ_a. To keep our notations neat, we shall drop the subscript from the terms s_a, P_a and Q_a.

THEOREM 1.4. *For every $a \in \mathbb{B}$, ϕ_a has the following properties:*

(i) $\phi_a(a) = 0$ *and* $\phi_a(0) = a$.
(ii) $\phi_a'(0) = -s^2 P - sQ$ *and* $\phi_a'(a) = -P/s^2 - Q/s$.
(iii) *The identity*

$$1 - \langle \phi_a(z), \phi_a(w)\rangle = \frac{(1 - \langle a, a\rangle)(1 - \langle z, w\rangle)}{(1 - \langle z, a\rangle)(1 - \langle a, w\rangle)}$$

holds for every $z \in \overline{\mathbb{B}}$.
(iv) *The identity*

$$1 - |\phi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \langle z, a\rangle|^2}$$

holds for every $z \in \overline{\mathbb{B}}$.
(v) ϕ_a *is an involution.*
(vi) ϕ_a *is a homeomorphism of* $\overline{\mathbb{B}}$ *onto* $\overline{\mathbb{B}}$, *and* $\phi_a \in Aut(\mathbb{B})$.

PROOF.

(i) Obvious from (1.2).
(ii) For $z \in \overline{\mathbb{B}}$, (1.2) can be written as

$$\begin{aligned}\phi_a(z) &= [1 + \langle z, a\rangle + \langle z, a\rangle^2 + \dots][a - (P + sQ)z] \\ &= \phi_a(0) + \langle z, a\rangle a - (P + sQ)z + O(|z|^2),\end{aligned}$$

and since $\langle z, a\rangle a = |a|^2 Pz$, the co-efficient of z in the above expansion is $(|a|^2 - 1)P - sQ$. This gives the first formula. The second formula follows from

$$\phi_a(a + h) = \frac{-Ph - sQh}{s^2 - \langle h, a\rangle}.$$

(iii) We split $\phi_a(z)$ into its components in $[a]$ and the one orthogonal to $[a]$ and substitute in the left side to get

$$1 - \frac{\langle a - Pz, a - Pw\rangle + s^2 \langle Qz, Qw\rangle}{(1 - \langle z, a\rangle)(1 - \langle a, w\rangle)}.$$

As P and Q are self-adjoint projections, we can replace Pw and Qw with w. Also,

$$\langle z, a\rangle \langle a, w\rangle = \langle \langle z, a\rangle a, w\rangle = |a|^2 \langle Pz, w\rangle .$$

From this the result follows.

(iv) Taking $z = w$ in the previous identity gives this identity. As an obvious consequence, we have $|\phi_a(z)| < 1$ iff $|z| < 1$. Thus, ϕ_a maps $\mathbb{B}$ into itself and S into itself.

(v) Set $\psi = \phi_a \circ \phi_a$. Then ψ is a holomorphic map of $\mathbb{B}$ into $\mathbb{B}$ with $\psi(0) = 0$, and from (2) we get

$$\psi'(0) = \phi_a'(a)\phi_a'(0) = P + Q = I.$$

By Proposition 1.1, $\psi \equiv I$ and we are done.

(vi) Clear from the previous parts.

□

The following consequence is trivial but is of great importance.

THEOREM 1.5. *$Aut(\mathbb{B})$ acts transitively on $\mathbb{B}$.*

Using these results and Proposition 1.3, we get the following theorem, which proves that there can be no Riemann mapping theorem when $n > 1$.

PROPOSITION 1.6. *Suppose Ω is a circular region in $\mathbb{C}^n$, $0 \in \Omega$, and some biholomorphic map F maps $\mathbb{B}$ onto Ω. Then there is a linear transformation of $\mathbb{C}^n$ that maps $\mathbb{B}$ onto Ω.*

PROOF. If $a = F^{-1}(0)$, then $F \circ \phi_a$ is a biholomorphic map of $\mathbb{B}$ onto Ω that fixes 0. Therefore, by Proposition 1.3, $F \circ \phi_a$ is linear. □

From this last result, it is clear that the ball and the polydisk cannot be biholomorphically equivalent, when $n > 1$.

THEOREM 1.7 (Poincaré's theorem). *When $n > 1$, there is no biholomorphic map of $\mathbb{B}$ onto the polydisk U^n.*

PROOF. Assume we have a biholomorphic map from $\mathbb{B}$ onto U^n. From the previous proposition, we would have a invertible linear map which takes $\mathbb{B}$ onto U^n. As invertible linear maps take balls onto ellipsoids, this is not possible. □

All unitary transformations are certainly automorphisms of the ball. We now show that the unitary transformations and the maps ϕ_a generate Aut($\mathbb{B}$).

THEOREM 1.8. *Let $\psi \in Aut(\mathbb{B})$ and $a = \psi^{-1}(0)$. Then, there is a unique unitary transformation U such that $\psi = U\phi_a$.*

PROOF. The map $\psi \circ \phi_a$ is an automorphism of $\mathbb{B}$ that fixes 0, and is hence a linear map by Proposition 1.3. As any linear map taking $\mathbb{B}$ onto $\mathbb{B}$ is unitary, there is an unique unitary transformation U such that $\psi \circ \phi_a = U$. As ϕ_a is an involution, this gives $\psi = U\phi_a$ as required. □

We now compute the automorphism group of the unit polydisk U^n. The polydisk is just a Cartesian product of the unit disk in $\mathbb{C}$. Therefore, applying an automorphism to each component yields an automorphism of U^n. Hence the result below is not surprising.

THEOREM 1.9. *Let $\psi \in Aut(U^n)$. Then, there exists a permutation $p : (1,\dots,n) \to (1,\dots,n)$, real numbers $\theta_1,\dots,\theta_n$, and complex numbers $\alpha_1,\dots,\alpha_n$, $|\alpha_j| < 1$ such that*

$$\psi(z) = \left(e^{i\theta_1}\frac{z_{p(1)} - \alpha_1}{1 - \overline{\alpha}_1 z_{p(1)}}, \dots, e^{i\theta_n}\frac{z_{p(n)} - \alpha_n}{1 - \overline{\alpha}_n z_{p(n)}}\right).$$

PROOF. Let $\alpha := \psi(0)$. Define

$$\sigma_\alpha(z) := \left(\frac{z_1 - \alpha_1}{1 - \overline{\alpha}_1 z_1}, \dots, \frac{z_n - \alpha_n}{1 - \overline{\alpha}_n z_n}.\right)$$

Then, $\sigma_\alpha \in Aut(U^n)$. Replacing ψ with $\sigma_\alpha \circ \psi$ we may assume $\psi(0) = 0$. It now suffices to prove that ψ is of the form

$$\psi(z) = (e^{i\theta_1} z_{p(1)}, \dots, e^{i\theta_n} z_{p(n)}).$$

By Proposition 1.3, ψ is linear and therefore writing ψ as $(\psi_1,\dots,\psi_n)$, we have

$$\psi_k(z) = \sum_{j=1}^{n} a_{kj} z_j, \;\; a_{kj} \in \mathbb{C}$$

For $r < 1$, let $z_j = re^{i\operatorname{Arg}(a_{kj})}$, $j = 1, \dots, n$. We have $|\psi_k(z)| < 1$, $k = 1, \dots, n$ and hence $\sum_{j=1}^n |a_{kj}| < 1/r$. As r is arbitrary, we have

$$\sum_{j=1}^{n} |a_{kj}| \leq 1. \tag{1.3}$$

For a given j, consider the sequence $z_\nu = \left\{(0, \dots, 1 - \frac{1}{\nu}, \dots, 0)\right\}_{\nu \in \mathbb{N}}$ where we have $1 - \frac{1}{\nu}$ in the j-th place and 0 elsewhere. Every limit point of $\{\psi(z_\nu)\}_{\nu \in \mathbb{N}}$ is on ∂U^n. As

$$\psi(z_\nu) = (1 - \frac{1}{\nu})(a_{1j}, \dots, a_{nj}) \to (a_{1j}, \dots, a_{nj}),$$

we conclude that the latter point is in ∂U^n. Therefore,

$$\max_{k=1,\dots,n} |a_{kj}| = 1, \quad j = 1, \dots, n.$$

Let $k(1)$ be such that $|a_{k(1),1}| = 1$. By (1.3), $a_{k(1),j} = 0$ for $j = 1, \dots, n$. Let $k(2)$ be such that $|a_{k(2),2}| = 1$. Since $a_{k(1),2} = 0$ and $a_{k(2),j} = 0$, $j \neq 2$, we have $k(2) \neq k(1)$. Thus, if $k(j)$ is such that $|a_{k(j),j}| = 1$, then $(k(1), \dots, k(n))$ is a permutation of $(1, \dots, n)$ and $a_{k(j),i} = 0$ for $i \neq j$. Let p be the inverse the above permutation. Then, we have $\psi_k(z) = a_{k,p(k)} z_{p(k)}$, $|a_{k,p(k)}| = 1$. Combining this with the remarks at the beginning of the proof, we are done. $\square$

We have proved that the ball and polydisk are biholomorphically inequivalent. Now, we define the concept of a proper map, which is a weaker notion than biholomorphism. The Remmert-Stein theorem will show that there cannot be proper map from a general class of domains, polydisks forming a subclass, into the ball.

DEFINITION 1.10. Let X and Y be topological spaces. A continuous map $f : X \to X$. is said to be *proper* if $f^{-1}(K)$ is compact in X for every compact $H \subseteq Y$.

In the case where $F : \Omega_1 \to \Omega_2$ is a proper map, where Ω_1 and Ω_2 are regions in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively, this is equivalent to the requirement that for every sequence $\{z_i\}$ in Ω_1 that has no limit point in Ω_1, $\{F(z_i)\}$ has no limit point in Ω_2.

We now state the classical Remmert-Stein theorem which we had alluded to in the introduction. We were interested in investigating a generalization that subsumes the following theorem and yields other interesting corollaries which are not covered by the classical Remmert-Stein theorem. Hence, we only state the theorem here. We shall present the proofs of a couple of generalizations of Theorem 1.11 in Chapter 2.

THEOREM 1.11 (Remmert-Stein [**6**]). *Let D be a domain in $\mathbb{C}^n$ with the property that $\exists p \in \partial D$, positive integers n_1, n_2, and and open neighbourhood U, $p \in U$, such that:*

(i) $U = U_1 \times U_2$.
(ii) *U_j is an open subset of $\mathbb{C}^{n_j}$, $j = 1, 2$, where $n_1 + n_2 = n$.*

(iii) $U \cap D = D_1 \times D_2$, D_j *domains in* $\mathbb{C}^{n_j}$, $j = 1, 2$, *with* $\overline{D}_2 \cap U_2 \neq U_2$.

Then, there is no proper holomorphic map from D *into* $\mathbb{B}_m$, $m > 0$.

1.2. Automorphisms of bounded domains

We now investigate some general results about automorphisms of a bounded domain. First, we state some standard results, which will be used in the proof of the main theorem due to Cartan. The proofs can be found in Chapter 5 of [**5**]

THEOREM 1.12 (Montel). *Let* $\mathcal{F}$ *be a family of of holomorphic functions on a domain* $\Omega \subseteq \mathbb{C}^n$ *such that, for any compact set* $K \subseteq \Omega$, *there exists* $M_K > 0$ *satisfying* $|f(z)| < M_K$ *for* $z \in K$, $f \in \mathcal{F}$. *Then any sequence* $\{f_\nu\}_{\nu \in \mathbb{N}}$, $f_\nu \in \mathcal{F}$, *contains a subsequence which converges uniformly on compact subsets of* Ω.

PROPOSITION 1.13. *Let* $\{\phi_\nu\}$ *be a sequence of continuous open mappings of* $\Omega \subseteq \mathbb{C}^n$ *into* $\mathbb{C}^n$. *Suppose that* ϕ_ν *converges uniformly on compact subsets of* Ω *to a map* $\phi : \Omega \to \mathbb{C}^n$. *Suppose that some* $a \in \Omega$ *is an isolated point of* $\phi^{-1}(\phi(a))$. *Then, for any neighbourhood of* U *of* a, *there is a* ν_0 *such that* $\phi(a) \in \phi_\nu(U)$ *for* $\nu \geq \nu_0$.

THEOREM 1.14 (Hurwitz). *Let* Ω *be a domain in* $\mathbb{C}^n$ *and* $\{f_\nu\}_{\nu \in \mathbb{N}}$ *be a sequence of holomorphic functions on* Ω, *which converge uniformly on compact subsets of* Ω *to a non-constant holomorphic function* f. *Then if* $f_\nu(z) \neq 0$, $\forall \nu \in \mathbb{N}$, *and all* $z \in \Omega$, *respectively, we have* $f(z) \neq 0$, $\forall z \in \Omega$.

THEOREM 1.15 (Cartan). *Let* D *be a bounded domain in* $\mathbb{C}^n$ *and let* $\{f_\nu\}_{\nu \in \mathbb{N}}$ *be a sequence of automorphisms of* D. *Suppose that* f_ν *converges uniformly on compact subsets of* D *to a holomorphic map* $f : D \to \mathbb{C}^n$. *Then, the following properties are equivalent.*

(i) $f \in Aut(D)$.

(ii) $f(D) \nsubseteq \partial D$.

(iii) *There exists* $a \in D$ *such that the complex Jacobian* $J_{\mathbb{C}}(f)(a)$ *is non-zero.*

PROOF.

(i) $\Rightarrow$ (ii) This is obvious.

(i) $\Rightarrow$ (iii) If $f \in Aut(D)$, then f is invertible and $f \circ f^{-1} \equiv I$ and hence

$$J_{\mathbb{C}}(f)(a) J_{\mathbb{C}}(f^{-1})(f(a)) = 1, \ a \in D$$

.

(iii) $\Rightarrow$ (ii) This is a straightforward consequence of the inverse function theorem for holomorphic mappings.

(ii) $\Rightarrow$ (iii) Clearly $f(D) \subseteq \overline{D}$ and therefore by (ii), there exists $a \in D$ such that $f(a) = b \in D$. Let $g_\nu = f_\nu^{-1}$. By Montel's theorem, we may, by passing to a subsequence, assume that $\{g_\nu\}$ converges uniformly on compact subsets of D to a holomorphic map $g : D \to \mathbb{C}^n$. We have $g(b) = \lim_{\nu\to\infty} f_\nu^{-1}(f(a))$. For large ν, $f_\nu(a)$ is close to $f(a)$ and hence lies in a compact subset of D. As g_ν converges uniformly on compact subsets of D, we get that

$$g(b) = \lim_{\nu\to\infty} g_\nu(f_\nu(a)) = a$$

Let V be a small neighbourhood of b such that $g(V)$ lies in a compact subset of D. Therefore, there is a compact subset K of D, such that for large ν, $g_\nu(V) \subseteq K$. Then, for $z \in V$, we have

$$f(g(z) = \lim_{\nu\to\infty} f_\nu(g_\nu(z)) = z.$$

Hence, $f \circ g = I$ on V and therefore the Jacobian of f is non-zero on points of $g(V)$.

(iii) $\Rightarrow$ (i) Each $J_{\mathbb{C}}(f_\nu)$ is holomorphic on D and converges uniformly on compact subsets of D to $J_{\mathbb{C}}(f)$. By hypothesis, $J_{\mathbb{C}}(f) \not\equiv 0$. As each f_ν is an automorphism, $J_{\mathbb{C}}(f_\nu(x)) \neq 0$ for $x \in D$. If $J_{\mathbb{C}}(f)$ is non-constant, then by Hurwitz's theorem it is never zero. If $J_{\mathbb{C}}(f)$ is constant then obviously it is never zero. Therefore, by the inverse function theorem for holomorphic mappings, f is an open map and any $x \in D$ is isolated in $f^{-1}f(x)$. By Proposition 1.13, we have $f(D) \subseteq \bigcup f_\nu(D) = D$. As in the previous part, let $g_\nu = f_\nu^{-1}$ and, without loss of generality, let g_ν converge to g uniformly on compact subsets of D. By repeating the argument in the previous part, we have $g(f(x)) = x,\ \forall x \in D$. Therefore, $J_{\mathbb{C}}(g)(y) \neq 0$ for $y \in f(D)$, and repeating the argument at the beginning of this part, we conclude $g(D) \subseteq D$. Hence, $f(g(x)) = x,\ \forall x \in D$. Therefore, $f \in Aut(D)$.

□

Given a bounded domain in $\mathbb{C}^n$, it is easy to see that the set of automorphisms, equipped with the compact-open topology, becomes a topological group. We now deduce an important result about the automorphism group using the above theorem. First we define the notion of a proper action.

DEFINITION 1.16. Let G be a locally compact topological group and X be a locally-compact Hausdorff topological space. Let G act continuously on X. We say that the action of G on X is proper, if the map $G \times X \to X \times X$ defined by $(g, x) \mapsto (gx, x)$, is proper.

THEOREM 1.17. *Let D be a bounded domain in $\mathbb{C}^n$. The action of $Aut(D)$ on D is proper.*

PROOF. Define

$$G(K, L) := \{f \in Aut(D) : f(K) \cap L \neq \emptyset\}.$$

We will prove that $G(K, L)$ is compact. Suppose, for the moment, that this is true. Then, $Aut(D)$ is locally compact (take $K \subseteq \mathrm{Int}(L)$). Any compact set in $X \times X$ is contained in a set of the form

$K \times K$, $K \subseteq X$ compact. The inverse image of $K \times K$ under the map $(g, x) \mapsto (gx, x)$ is just $G(K, K)$ and is hence compact.

Now, let $\{f_\nu\}_{\nu \in \mathbb{N}}$ be a sequence of elements of $G(K, L)$. By Montel's theorem, we may assume that f_ν converges uniformly on compact subset of D to a holomorphic map f. Since $f_\nu(K) \cap L \neq \emptyset$, we can find elements $a_\nu \in K$, $f_\nu(a_\nu) = b_\nu \in L$. As K and L are compact, by passing to subsequences, we may assume $a_\nu \to a \in K$, $b_\nu \to b \in L$. Then $f(a) = b$, and by Theorem 1.15, $f \in Aut(D)$. Hence, $G(K, L)$ is compact and we are done. This establishes that the action of $Aut(D)$ on D is proper.

□

The above result is a key step in establishing, in conjunction with several ideas in Lie theory, the following deep result.

THEOREM 1.18 (Cartan). *Let D be a bounded domain in $\mathbb{C}^n$. Then $Aut(D)$ is a Lie group whose dimension $\leq n^2 + n$.*

CHAPTER 2

Some Generalizations of the Remmert-Stein Theorem

Recall the theorem of Remmert-Stein (Theorem 1.11) that we stated in Chapter 1. In the course of this project, we found that we could state a more generalized result Theorem 2.1 below, from which the Remmert-Stein theorem follows. A consequence of Theorem 2.1 is Corollary 2.2, which is a version of the classical Remmert-Stein Theorem, with the ball replaced by a strictly pseudoconvex domain. We then present Theorem 2.4 , where the condition $\overline{D_j} \cap U_j \neq U_j$ fails for $j = 1, 2$. We shall conclude this Chapter with an example.

THEOREM 2.1. *Let Ω be a bounded domain in $\mathbb{C}^m$ such that $\partial\Omega$ contains no germs of non-trivial complex-analytic curves. Let D be a domain in $\mathbb{C}^n$ with non-smooth boundary with the property that $\exists p \in \partial D$, positive integers n_1, n_2, and an open neighbourhood U, $p \in U$ such that :*

(i) $U = U_1 \times U_2$.

(ii) *U_j is an open subset of $\mathbb{C}^{n_j}$, $j = 1, 2$, where $n_1 + n_2 = n$.*

(iii) *$U \cap D = D_1 \times D_2$, D_j domains in $\mathbb{C}^{n_j}$, $j = 1, 2$, with $\overline{D_2} \cap U_2 \neq U_2$.*

Then, there is no proper holomorphic map from D into Ω.

PROOF. Let $f = (f_1, \dots, f_m)$ be a proper holomorphic map of D into Ω. Our strategy is to arrive at a contradiction. Write z as (ξ, ω), $\xi \in \mathbb{C}^{n_1}$ and $\omega \in \mathbb{C}^{n_2}$. Let $\omega_\nu \in D_2$ and suppose $\omega_\nu \to \omega \in (\partial D_2) \cap U_2$. For $j = 1, \dots, m$, the functions $\xi \mapsto f_j(z, \omega_\nu)$ define holomorphic functions $\phi_{j,\nu}$ on D_1, with $\sum |\phi_{j,\nu}|^2$ bounded. By Montel's theorem, we may assume, by passing to a subsequence, that $\phi_{j,\nu} \to \phi_j$ uniformly on compact subsets of D_1. For any $\xi \in D_1, \{(\xi, \omega_\nu)\}_{\nu\in\mathbb{N}}$ has no limit point in D. Since f is proper, $\{f(\xi, \omega_\nu)\}_{\nu\in\mathbb{N}}$, where $f(\xi, \omega_\nu) = (\phi_{1,\nu}(\xi), \dots, \phi_{m,\nu}(\xi))$, has no limit point in Ω. Hence, $\Phi := (\phi_1(\xi), \dots, \phi_m(\xi)) \in \partial\Omega\ \forall\xi \in D_1$. As D_1 does not contain any germs of non-trivial complex-analytic curves, it does not contain non-trivial analytic disks. Therefore, for each connected component of $D_1 \cap \Lambda$, say $\mathcal{K}_\alpha(\Lambda)$, where Λ is a complex line in $\mathbb{C}^{n_1}$, $\Phi_{|\mathcal{K}_\alpha(\Lambda)} \equiv$ constant. This proves that $\frac{\partial\phi_j}{\partial\xi_p} \equiv 0$ $(p = 1, \dots, n_1)$. Now, by Weierstrass' theorem, defining $\xi := (\xi_1, \dots, \xi_{n_1})$, we conclude that

$$\frac{\partial f_j(\xi, \omega_\nu)}{\partial\xi_p} \longrightarrow \frac{\partial\phi_j}{\partial\xi_p} = 0.$$

Hence, for $p = 1, \dots, n_1$, $\frac{\partial f_j(\xi,\omega_\nu)}{\partial\xi_p}$ tends to 0 if ω tends to a point of $(\partial D_2) \cap U_2$. Therefore, for fixed $\xi \in D_1$, the function

$$\omega \longmapsto \begin{cases} \frac{\partial f_j(\xi,\omega)}{\partial \xi_p} & \text{if } \omega \in D_2 \cap U_2, \\ 0 & \text{if } \omega \in U_2 \setminus D_2. \end{cases}$$

is holomoprhic on U_2 by Rado's theroem. As $U_2 - \overline{D}_2$ is non-empty, we conclude that $\frac{\partial f_j(\xi,\omega)}{\partial \xi_p)} \equiv 0$ on $D_1 \times D_2,\ p = 1, \ldots, n_1$. Hence, $\frac{\partial f_j}{\partial \xi_p} \equiv 0$ on $D,\ p = 1, \ldots, n_1$. Therefore, the map f is constant on any connected component of $D_1 \times \{\omega_0\}$, $\omega_0 \in D_2$, and $D_1 \times \{\omega_0\}$ is not compact in D. Hence, $f^{-1}\{f(\xi_0, \omega_0)\}$ is non-compact when $(\xi_0, \omega_0) \in D_1 \times D_2$, which contradicts the properness of f.

□

From Theorem 2.1, it is clear that the classical Remmert-Stein Theorem, with the ball replaced by a strictly pseudo-convex domain, will follow if we could prove that a strictly pseudo-convex domain contains no non-trivial analytic disks on the boundary.

COROLLARY 2.2. *The conclusion of Theorem 1.11 still holds, if the ball is replaced by a strictly pseudo-convex domain* Ω *in* $\mathbb{C}^m$.

PROOF. Let ρ be a defining function for Ω. Let $\Phi = (\phi_1, \ldots, \phi_m)$ be an analytic disk on the boundary of Ω. Then $\rho(\phi_1, \ldots, \phi_m) \equiv 0$. A straightforward computation reveals that,

$$\left[\frac{\partial \Phi}{\partial z_\mu}\right]^T \mathcal{H}_{\mathbb{C}}(\Phi(\rho)) \left[\frac{\partial \overline{\Phi}}{\partial z_\mu}\right] \equiv 0.$$

By the strict pseudo-convexity of Ω, we must have $\frac{\partial \Phi}{\partial z_\mu} \equiv 0$. Therefore, Φ is a trivial analytic disk. □

In the proof of Theorem 2.1, we had used the fact that $U_2 \setminus \overline{D_2}$ is a non-empty open set and hence a *set of uniqueness* to conclude that $\frac{\partial f_j(\xi,\omega)}{\partial \xi_p)} \equiv 0$ on $D_1 \times D_2,\ p = 1, \ldots, n_1$. Lemma 2.3 shows the existence of sets of uniqueness which are, in some sense, thinner than open sets.

LEMMA 2.3. *Let* U *be a domain in* $\mathbb{C}^n$ *and let* V *be a real n-dimensional sub-manifold of* U *such that,* $T_p(V) \cap iT_p(V) = 0, \forall p \in T_p(V)$*, i.e.* $T_p(V)$ *has no non-trivial complex sub-space. Then, for every* $f \in \mathcal{O}(U)$ *such that,* $f \equiv 0$ *on* V*, we have* $f \equiv 0$ *on* U*, i.e.* V *is a set of uniqueness.*

PROOF. Let $p \in V$ and let $L_1 : T_p(V) \to \mathbb{R}^n$ be a real isomorphism. Define $L : \mathbb{C}^n \to \mathbb{C}^n$ as $z \mapsto L_1(\pi_p(z)) + iL_1(-i(z - \pi_p(z)))$. As $T_p(V) \oplus iT_p(V) = \mathbb{C}^n$, any $z \in \mathbb{C}^n$ can be written as $z = x + iy, x, y \in T_p(V)$, in an unique way; and we may define $\pi_p(z) := x$. Hence, the map L is well-defined and is a complex isomorphism with $L(T_p(V)) = \mathbb{R}^n$. So, we can assume, without loss of generality, that $T_p(V) = \mathbb{R}^n$. But, $f \equiv 0$ on V and therefore all real derivatives of f vanish at p. Hence, all first order complex derivatives of f vanish at p. Since $p \in V$

was arbitrary, we can repeat the argument with $\frac{\partial f}{\partial z_j}$ $(j = 1, \ldots, n)$ in place of f. Proceeding inductively, we conclude that at a point $p \in V$, all derivatives of all orders vanish. Hence, $f \equiv 0$ on U as required. $\square$

We now exploit Lemma 2.3 to prove a generalization of the Remmert-Stein theorem, in which the set $D_2 \setminus U_2$ needs to contain only a maximally totally-real submanifold.

THEOREM 2.4. *Let D be a domain in $\mathbb{C}^n$ with non-smooth boundary with the property that $\exists p \in \partial D$, positive integers n_1, n_2 and an open neighbourhood U, $p \in U$ such that :*

(i) $U = U_1 \times U_2$.

(ii) U_j *is an open subset of* $\mathbb{C}^{n_j}$, $j = 1, 2$, *where* $n_1 + n_2 = n$.

(iii) $U \cap D = D_1 \times D_2$, D_j *domains in* $\mathbb{C}^{n_j}$, $j = 1, 2$, *with* $\overline{D}_2 \cap U_2 = U_2$.

Suppose there exists a subdomain $V_0 \subseteq U_2$ such that $U_2 \setminus D_2$ contains a maximally totally-real submanifold of V_0, then there is no proper holomorphic map from D into $\mathbb{B}_m$.

PROOF. Let $f = (f_1, \ldots, f_m)$ be a proper holomorphic map of D into $\mathbb{B}_m$. Write z as (ξ, ω), $\xi \in \mathbb{C}^{n_1}$ and $\omega \in \mathbb{C}^{n_2}$. Let $\omega_\nu \in D_2$ and suppose $\omega_\nu \to \omega \in (\partial D_2) \cap U_2$. For $j = 1, \ldots, m$, the functions $\xi \mapsto f_j(z, \omega_\nu)$ define holomorphic functions $\phi_{j,\nu}$ on D_1 with $\sum |\phi_{j,\nu}|^2$ bounded. By Montel's theorem, we may assume by passing to a sub-sequence that $\phi_{j,\nu} \to \phi_j$ uniformly on compact subsets of D_1. For any $\xi \in D_1, \{(\xi, \omega_\nu)\}_{\nu \in \mathbb{N}}$ has no limit point in D. Since f is proper, $\{f(\xi, \omega_\nu)\}_{\nu \in \mathbb{N}}$, where $f(\xi, \omega_\nu) = (\phi_{1,\nu}(\xi), \ldots, \phi_{m,\nu}(\xi))$, has no limit point in $\mathbb{B}_m$. Hence, $\Phi := (\phi_1(\xi), \ldots, \phi_m(\xi)) \in \partial \mathbb{B}_m$ $\forall \xi \in D_1$. Therefore, $|\Phi(\xi)| = 1,\ \forall \xi \in D_1$. As a result, each $\phi_j \equiv$ constant, $j = 1, \ldots, m$. This proves that $\frac{\partial \phi_j}{\partial \xi_p} \equiv 0$ $(p = 1, \ldots, n_1)$.

Now, by Weierstrass' theorem, defining $\xi := (\xi_1, \ldots, \xi_{n_1})$, we conclude

$$\frac{\partial f_j(\xi, \omega_\nu)}{\partial \xi_p} \to \frac{\partial \phi_j}{\partial \xi_p} = 0.$$

Hence, for $p = 1, \ldots, n_1$, $\frac{\partial f_j(\xi, \omega_\nu)}{\partial \xi_p}$ tends to 0 if ω tends to a point of $(\partial D_2) \cap U_2$. Hence, for fixed $\xi \in D_1$, the function

$$\omega \longmapsto \begin{cases} \frac{\partial f_j(\xi, \omega)}{\partial \xi_p} & \text{if } \omega \in U_2 \cap D_2, \\ 0 & \text{if } \omega \in U_2 \setminus D_2. \end{cases}$$

is holomoprhic on U_2 by Rado's theroem. As $V_0 \subseteq U_2$ contains a maximal totally-real submanifold on which the above function vanishes, we conclude $\frac{\partial f_j(\xi, \omega)}{\partial \xi_p)} \equiv 0$ on $D_1 \times D_2$, $p = 1, \ldots, n_1$. Since D is connected, $\frac{\partial f_j}{\partial \xi_p} \equiv 0$ on D, $p = 1, \ldots, n_1$. Arguing exactly as in the end of the proof of Theorem 2.1, we are done.

$\square$

It would be instructive to see an example of the sort of domain considered in Theorem 2.4.

EXAMPLE 2.5. Write $(z, w) \in \mathbb{C}^2$ as $(x + iy, u + iv)$. Let $D := \mathbb{B} \setminus V$, where $V := \{(x, y) : |x| < 1/100,\ |y| < 1/100\}$. Let U be a polydisk centred at the origin with each component disk being the unit disk in $\mathbb{C}$. Then $U \cap D$ is

$$\mathbb{D} \setminus \{(x, 0) : |x| < 1/100\} \times \mathbb{D} \setminus M,$$

where $M := \{(u, 0) : |u| < 1/100\}$. Then, M is a maximally totally real sub-manifold of U_2 and hence the hypothesis of Theorem 2.4 is satisfied at $p = (0, 0)$. The hypothesis of the classical Remmert-Stein Theorem are not satisfied at any point of ∂D (there is no loss of generality in assuming $p = 0$ and U is a polydisk).

CHAPTER 3

Analysis of Proper Maps

In this chapter, we study the structure of proper maps. We begin by proving the Two Function Lemma, of which the Weierstrass Preparation Theorem is a simple corollary. Then, we use the Preparation theorem to derive some interesting results on analytic subvarieties. In the next section, we use the results on analytic varieties to study the structure of proper maps. The main result we prove is that, a proper map, between domains of the same dimension, has associated with it a "multiplicity", which gives the number of pre-images of a regular value. The image of a subvariety under such a map is also a subvariety. These results will be used in the next chapter, and is vital in the study of automorphisms and proper maps.

3.1. Analytic Varieties

We first generalize the definition of the order of a zero.

DEFINITION 3.1. Suppose Ω is a domain in $\mathbb{C}^n$, $f \in \mathcal{O}(\Omega)$, $a \in \Omega$ and $f(a) = 0$. If f is not identically zero, then there are vectors $b \in \mathbb{C}^n$ such that the one-variable function $\lambda \mapsto f(a + \lambda b)$ does not vanish identically in any neighbourhood of $\lambda = 0$; and hence has a zero of positive integral order k at $\lambda = 0$. The smallest k which can be obtained by varying b is called the *order of the zero* of f at a.

If f has a zero of order m at 0, we may choose co-ordinates so that $f(0', z_n)$ has a zero of order m at $z_n = 0$, $z' \in \mathbb{C}^{n-1}$, $z_n \in \mathbb{C}$. We also write polydisks in $\mathbb{C}^n$ in the form, $\Delta = \Delta' \times \Delta_n$, where Δ' is a polydisk in $\mathbb{C}^{n-1}$ and Δ_n is a disk in $\mathbb{C}$. We assume $n > 1$ for the rest of this section.

LEMMA 3.2 (Two-Function Lemma). *Suppose Ω is neighbourhood of 0 in $\mathbb{C}^n$, $f \in \mathcal{O}(\Omega)$, $g \in \mathcal{O}(\Omega)$, and $f(0', z_n)$ has a zero of multiplicity m at $z_n = 0$. Then:*

(i) *There is a polydisk $\Delta = \Delta' \times \Delta_n \subseteq \Omega$, with centre at 0, such that $f(z', \cdot)$ has, for each $z' \in \Delta'$, exactly m zeros in Δ_n, counted with multiplicity.*

(ii) *If these zeros are denoted by $\alpha_1(z'), \dots, \alpha_m(z')$, then the elementary symmetric functions of the unordered m-tuple*

$$\{g(z', \alpha_j(z')) : 1 \leq j \leq m\},$$

are holomorphic functions in Δ'.

PROOF.

(i) Since the zeros of a holomorphic function in one-variable are isolated and f has a zero of order m at 0, we can choose a disk Δ_n of radius r around the origin such that, $f(0',\cdot)$ has no zero in $\overline{\Delta_n} \setminus \{0\}$. Hence, there exists a $\delta > 0$, and a polydisk Δ' in $\mathbb{C}^n$, centred at $0'$, such that $|f(z',\lambda)| > \delta$ whenever $z' \in \Delta'$ and $|\lambda| = r$, and such that the closure of $\Delta := \Delta' \times \Delta_n$ lies in Ω. For every $h \in \mathcal{O}(\Omega)$ and $z' \in \Delta'$, we define

$$J_h(z') := \frac{1}{2\pi i} \int_{|\lambda|=r} \left(\frac{h D_n f}{f} \right) (z', \lambda) d\lambda,$$

where $D_n f = \frac{\partial f}{\partial z_n}$. The denominator is bounded away from zero on the path of integration. Thus, J_h is continuous in Δ', and by Morera's theorem, we see that, $f \in \mathcal{O}(\Delta')$. By the residue theorem, when $h \equiv 1$, $J_h(z')$ counts the number of zeroes of $f(z',\cdot)$ in Δ_n. Therefore, $J_h(0') = m$. Being a continuous integer-valued function in the connected set Δ', J_h is constant.

(ii) For an arbitrary h, by the residue theorem, we have

$$J_h(z') = \sum_{j=1}^{m} h(z', \alpha_j(z')) \tag{3.1}$$

where $\alpha_j(z')'s$ are the zeros of $f(z',\cdot)$ in Δ_n. Therefore, the sum in (3.1) is holomorphic in Δ'. If $\xi \in \mathbb{C}$ and $|\xi|$ is sufficiently small, then $|\xi g| < 1$ on some neighbourhood of Δ', and the reasoning above can be applied to $h = log(1 - \xi g)$. Substituting for h in the sum (3.1), and exponentiating, we see that $G_\xi \in \mathcal{O}(\Delta')$, where

$$G'_\xi := \prod_{j=1}^{m} \left[1 - \xi g(z', \alpha_j(z')) \right].$$

For each $z' \in \Delta'$, $G_\xi(z')$ is a polynomial in ξ. The coefficient of ξ^k is

$$\frac{1}{2\pi i} \int_\Gamma G_\xi(z') \xi^{-k-1} d\xi,$$

where is Γ is a small circle around the origin. Since $G_\xi(z') \in \mathcal{O}(\Delta'))$, the above integral defines a holomorphic function in Δ'. The coefficients ξ^k are, upto a sign, precisely the elementary symmetric functions of $g(z'), \alpha_j(z'),\ j = 1, \ldots, m$.

$\square$

THEOREM 3.3 (Weierstrass Preparation Theorem). *Suppose Ω is neighbourhood of 0 in $\mathbb{C}^n$, $f \in \mathcal{O}(\Omega)$, $f(0', z_n)$ has a zero of multiplicity m at $z_n = 0$, and $\Delta = \Delta' \times \Delta_n$ is as before. Then there exists $h \in \mathcal{O}(\Delta)$, h has no zero in Δ, and*

$$W(z) = z_n^m + b_1(z') z_n^{m-1} + \cdots + b_m(z'),$$

with $b_j \in \mathcal{O}(\Delta'),\ b_j(0') = 0$*, such that*

$$f(z) = W(z)h(z) \ \ (z \in \Delta).$$

PROOF. If we apply the Two-Function Lemma with $g(z) = z_n$, and if we define

$$W(z) = W(z', z_n) := \prod_{j=1}^{m} \left[z_n - \alpha_j(z') \right],$$

we see that W is a monic polynomial in z_n whose coefficients b_j are holomorphic in Δ'. Also, $\alpha_j(0') = 0$, for $1 \le j \le m$. Thus, $W(0', z_n) = z_n^m$, and $b_j(0') = 0$. We define

$$h(z) = \frac{1}{2\pi i} \int_{|\lambda|=r} \left(\frac{f}{W} \right)(z', \lambda) \frac{d\lambda}{\lambda - z_n} \ \ (z \in \Delta),$$

where r is as in the proof of the two-function lemma. W has no zeros on the path of integration and $W(z', \cdot\,)$ is a polynomial with the same zeros as $f(z', \cdot\,)$. Therefore, $h \in \mathcal{O}(\Delta)$, and Riemann's removable singularities theorem shows that $f = Wh$ in Δ. □

We now define the important notion of an analytic subvariety.

DEFINITION 3.4. Let Ω be an open set in $\mathbb{C}^n$. A set $V \subseteq \Omega$ is said to be an *analytic subvariety* of Ω if :

(i) V is closed in Ω, and
(ii) every point $p \in \Omega$ has a neighbourhood $N(p)$ such that

$$V \cap N(p) = Z(f_1) \cap \cdots \cap Z(f_r)$$

for some $f_1, \ldots, f_r \in \mathcal{O}(N(p))$, where $Z(f)$ denotes the zero set of f. We call $Z(f)$ the *zero-variety* of f.

EXAMPLE 3.5. (i) Let Ω be a domain in $\mathbb{C}^n$. The empty set and Ω are subvarieties of Ω.
(ii) If Ω is a domain in $\mathbb{C}$ then the only non-trivial subvarieties of Ω are the discrete subsets.
(iii) Finite unions and intersections of subvarieties are also subvarieties.

We now prove the Projection Theorem. As before, we denote points in $\mathbb{C}^n$ as (z', z_n) and polydisks Δ as $\Delta' \times \Delta_n$. Let π denote the projection of $\mathbb{C}^n$ onto $\mathbb{C}^{n-1}$, $(z', z_n) \mapsto z'$.

THEOREM 3.6 (The Projection Theorem). *Let V be an analytic subvariety of a domain $\Omega \subseteq \mathbb{C}^n$, $n > 1$, let $p = (p', p_n)$ be a point of V, and let*

$$L := \{(p', \lambda) : \lambda \in \mathbb{C}\}.$$

If p is an isolated point of $L \cap V$, then p is the centre of a polydisk $\Delta \subseteq \Omega$ such that $\pi(V \cap \Delta)$ is an analytic subvariety of $\pi(\Delta)$.

PROOF. We may assume that p is the origin of $\mathbb{C}^n$ and that Ω is a polydisk in which V is defined by holomorphic functions $f_1, \ldots, f_r$. By hypothesis, there is a f_i, say f_r, for which the origin $\mathbb{C}$ is an isolated zero of $f_r(0', \cdot)$. For emphasis, we denote f_r by F.

There is a polydisk, $\Delta = \Delta' \times \Delta_n \subseteq \Omega$, with centre at 0, such that the conclusion of Two-Function Lemma holds for Δ, F, and any $g \in \mathcal{O}(\Omega)$. Therefore, the product P defined by

$$P(z') = \prod_{j=1}^{m} g(z', \alpha_j(z')) \ \ (z' \in \Delta'),$$

is holomorphic in Δ', where $\alpha_j(z')$, $(1 \leq j \leq m)$ are the zeros of $F(z', \cdot)$.

Fix $z' \in \Delta'$. It is clear that $P(z') = 0$ iff some $\alpha_j(z')$ is also a zero of $g(z', .)$ iff F and g have a common zero in Δ that lies in the pre-image of z' under π. Hence,

$$\pi(\Delta \cap Z(F) \cap Z(g)) = Z(P). \tag{3.2}$$

Since $P \in \mathcal{O}(\Delta')$, we conclude that $\pi(\Delta \cap Z(F) \cap Z(g))$ is an analytic subvariety of Δ'. We may assume that $r > 1$. Let (c_{ij}) be a rectangular matrix of complex numbers, with $(r-1)m$ rows and $(r-1)$ columns in which every square submatrix of size $(r-1) \times (r-1)$ has non-zero determinant. Define

$$g_i := \sum_{j=1}^{r-1} c_{ij} f_j \ \ (1 \leq i \leq rm - m),$$

Applying (3.2) to g_i in place of g, we see that each of the sets $E_i := \pi(\Delta \cap Z(F) \cap Z(g_i))$ is a subvariety of Δ'. We claim that

$$\pi(\Delta \cap V) = \bigcap_i E_i. \tag{3.3}$$

Let $z \in \Delta \cap V$. Then $z \in Z(g_i)$ for each i, and $z \in Z(F)$. Hence, $\pi(z) \in E_i$ for each i. Thus, the left side of (3.3) is a subset of the right. For the opposite inclusion, let $z' \in \bigcap E_i$. To each of the $(r-1)m$ values of i, there corresponds an $\alpha_k(z')$ such that $g_i(z', \alpha_k(z')) = 0$. Since k runs over only m values, there is some k and some set I of $r-1$ distinct i's, for which $g_i(z', \alpha_k(z')) = 0$. Therefore, the corresponding system of equations

$$\sum_{j=1}^{r-1} c_{ij} f_j(z', \alpha_k(z')) = g_i(z', \alpha_k(z')) = 0 \ \ (i \in I)$$

has a unique solution, by our choice of (c_{ij}). Thus, $f_j(z', \alpha_k(z')) = 0$ for all j and therefore $z' \in \pi(\Delta \cap V)$, proving our claim. Since each E_i is a subvariety of Δ', the same is true of their intersection and we are done. $\square$

We now use the Projection Theorem to prove the finiteness theorem. The finiteness theorem is used in the next section to prove that a proper map cannot "decrease dimension".

THEOREM 3.7. *Every compact analytic subvariety of $\mathbb{C}^n$ is a finite set of points.*

PROOF. The proof is by induction on the dimension. When $n = 1$, the result follows from the fact that the zero set of a non-trivial analytic function of one variable is discrete. Assume that $n > 1$ and that the theorem is true for $\mathbb{C}^{n-1}$. Let V be a compact subvariety of $\mathbb{C}^n$. Pick $z' \in \pi(V)$, and define

$$L := \{(z', \lambda) : \lambda \in \mathbb{C}\}.$$

We identify L with $\mathbb{C}$. We see that $L \cap V$ is a compact subvariety of $\mathbb{C}$ and is hence finite. Let p_i $(1 \leq i \leq m)$ be points of $L \cap V$. By the projection theorem, each p_i is the centre of a polydisk $\Delta_i \subseteq \mathbb{C}^n$ such that $\pi(V \cap \Delta_i)$ is a subvariety of $\pi(\Delta_i)$. The part of V that is not covered by $\Delta_1 \cup \ldots \Delta_m$ is compact, and hence has a positive distance from L. Hence, z' is the centre of a polydisk

$$\Delta' \subseteq \pi(\Delta_1) \cap \cdots \cap \pi(\Delta_m),$$

so small that all points of V that project into Δ' lie in $\Delta_1 \cup \ldots \Delta_m$. Therefore,

$$\Delta' \cap \pi(V) = \Delta' \cap \bigcup_{i=1}^{m} \pi(V \cup \Delta_i).$$

Thus, $\Delta' \cup \pi(V)$ is a sub-variety of Δ'. Since Δ' is a neighbourhood of the arbitrarily chosen point $z' \in \pi(V)$, and since $\pi(V)$ is compact, it follows that $\pi(V)$ is a subvariety of $\mathbb{C}^{n-1}$. Hence, by our induction hypothesis, $\pi(V)$ is a finite set. Since each point of $\pi(V)$ is the image of only finitely many points of V under π, we conclude that V is the union of finitely many finite sets. □

3.2. Structure of proper holomorphic maps

We now use the results on analytic varieties to study the structure of proper maps. Let Ω_1 and Ω_2 be domains in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively, and suppose $F : \Omega_1 \to \Omega_2$ is a proper holomorphic map. If $w = (w_1, \ldots, w_m) \in \Omega_2$, then $F^{-1}(w)$ is a subvariety of Ω_1, being the intersection of the zero sets of the functions $f_i - w_i$, where f_i is the i-th component of F. Since $F^{-1}(w)$ is compact by the properness of F, Theorem 3.7 shows that $F^{-1}(w)$ is finite. The number of inverse images will be denoted by $\#(w)$.

THEOREM 3.8. *When $m < n$, there cannot be a proper holomorphic map from Ω_1 into Ω_2.*

PROOF. Suppose F is a proper holomorphic map from Ω_1 into Ω_2. Let DF denote the real derivative of F, viewing F as a map from $\Omega_2 \subset \mathbb{R}^{2n}$ into $\mathbb{R}^{2m}$. The operator DF cannot have rank less than $2n$ at all points of Ω_1. Otherwise, the rank of the linear operator DF would be at most $2m-1$, and hence by the rank theorem, there is a $w \in \Omega_2$ such that $F^{-1}(w)$ contains a 1-dimensional manifold and hence an infinite set. Therefore, $n \geq m$. □

We now study the most interesting case, $m = n$. We define the terms *critical value* and *regular value* as usual. We call the set of all critical values the *critical set*. It is easy to see that F is a closed map. We summarize, without proof, some simple results which we need for the main theorem. The proofs can be found in Chapter 15 of [**7**].

PROPOSITION 3.9. *With notation as above, we have :*

(i) $F(\Omega_1) = \Omega_2$ *and*
(ii) *the regular values of F form a connected open set that is dense in Ω_1.*

In the next result, we *do not* assume that F is proper.

PROPOSITION 3.10. *Suppose Ω is a domain in $\mathbb{C}^n$, $F : \Omega \to \mathbb{C}^n$ is holomorphic map, and $F^{-1}(w)$ is compact for every $w \in \mathbb{C}^n$. Then every neighbourhood of any $p \in \Omega$ contains a connected neighbourhood D of p such that the restriction of F to D is a proper map of D onto $F(D)$. Consequently, F is an open map.*

The next theorem is a direct generalization of a well known result for holomorphic functions of one variable.

THEOREM 3.11 (Osgood's Theorem). *If Ω is a domain in $\mathbb{C}^n$ and $F : \Omega \to \mathbb{C}^n$ is holomorphic and injective, then F is a biholomorphism.*

We now come to the main structure theorems.

THEOREM 3.12. *Suppose Ω_1 and Ω_2 are domains in $\mathbb{C}^n$, and $F : \Omega_1 \to \Omega_2$ is a proper holomorphic map. Then there is an integer m such that $\#(w) = m$ when w is a regular value, and $\#(w) < m$ when w is a critical value .*

PROOF. Let $w_0 \in \Omega_2$, let $\#(w_0) = k,\ F^{-1}(w) = \{z_1, \dots, z_k\}$. There are open balls Q_i with centre z_i, whose closures are disjoint and lie in Ω_1. We set

$$E := \Omega_1 \setminus (Q_1 \cup \cdots \cup Q_k)\,.$$

As F is proper, F is a closed map, hence $F(E)$ is closed in Ω_2, so that w_0 is the centre of an open ball $N \subseteq \Omega_2 \setminus F(E)$. Define

$$D_i := Q_i \cap F^{-1}(N) \;\; (i = 1, \dots, k).$$

Let K be a compact subset of N. Since no boundary point of Q_i maps into N,

$$D_i \cap F^{-1}(K) = \overline{Q}_i \cap F^{-1}(K),$$

and the latter set is compact. Hence, $F : D_i \to N$ is proper, for each i. The restriction of F to any connected component Δ of D_i is proper and hence, by Proposition 3.9, $F(\Delta) = N$; but w_0 has only one inverse image in D_i and therefore each D_i is connected. Furthermore, F maps no pint outside $D_1 \cup \cdots \cup D_k$ into N, since N does not intersect $F(E)$.At this point, we have established the following,

($*$) **Fact.** If $w_0 \in \Omega_2$, $\#(w_0) = k,\ F^{-1}(w_0) = \{z_1, \dots, z_k\}$, then w_0 has a neighbourhood N and the z_i's have disjoint connected neighbourhoods D_i such that $F(D_i) = N$ for $1 \le i \le k$, and $F^{-1}(N) = D_1 \cup \dots D_k$. Moreover, the D_i's can be taken to lie in prescribed neighbourhoods of the points z_i.

Now let w_0 be a regular value of F. By the inverse function theorem, the D_i's can be chosen so that F is injective on each D_i. From ($*$), we see that, $\#(w) = \#(w_0)$ for each $w \in N$. Since the set of regular values is connected (Proposition 3.9), there is an m which satisfies the conclusion of the theorem for regular values. For an arbitrary $w_0 \in \Omega_2$, we see that N contains a regular value by Proposition 3.9. Hence, ($*$) implies that $\#(w_0) \le m$ for every $w_0 \in \Omega_2$. If $\#(w_0) = m$, then F is injective on each D_i. By Osgood's theorem, w_0 is a regular value of F. □

THEOREM 3.13. *With the same hypothesis as above, $F(V)$ is an analytic subvariety of Ω_2, whenever V is an analytic subvariety of Ω_1.*

PROOF. We first prove that the critical set of F is a zero-variety. Let w_0 be a regular value of F. The inverse function theorem and ($*$) show that there are holomorphic maps $p_i : N \to D_i$ $(1 \le i \le m)$ that invert F. Therefore, the product

$$\psi(w) := \prod_{i=1}^{m} J_{\mathbb{C}}(p_i(w))$$

is holomorphic in $\Omega_2 \setminus F(M)$, where $J_{\mathbb{C}}$ denotes the complex Jacobian and where $M = Z(J_{\mathbb{C}}(F))$, and has no zero in this region. For points in $F(M)$, we set ψ to be zero. If we could prove that ψ is continuous, then by Rado's theorem, $\psi \in \mathcal{O}(\Omega_2)$ and $F(M)$ would be a zero-variety. To this end, choose $w_0 \in F(M),\ z_1 \in M$ such that $F(z_1) = w_0$, and fix $\varepsilon > 0$. We apply ($*$) with the neighbourhood D_1 of z_1 chosen so small that, $|J| < \varepsilon$ in D_1. At least one factor in the definition of ψ is then of absolute value $< \varepsilon$ in N and the others are bounded in N proving that ψ is continuous at w_0.

For the general case, let $g \in \mathcal{O}(\Omega_1)$ and let maps $p_1, \dots, p_m$ be as before. The product

$$h(w) := \prod_{i=1}^{m} g(p_i(w)),$$

is then holomorphic in $\Omega_2 \setminus F(M)$. If K is compact in Ω_2, and $w \in K \setminus F(M)$, then $p_i(w)$ lies in the compact set $F^{-1}(K)$ for each i. Thus, h is bounded on $K \setminus F(M)$. By Riemann's removable singularities theorem, h extends to a holomorphic function on Ω_2 and we have $F(Z(g)) = Z(h)$, proving that $F(Z(g))$ is a subvariety of Ω_2.

Assume next that $V = Z(f_1) \cap \ldots Z(f_{r-1})$, where $f_1, \ldots, f_{r-1} \in \mathcal{O}(\Omega_1)$. Define

$$g_i := \sum_{j=1}^{r-1} c_{ij} f_j \quad (1 \leq i \leq rm - m),$$

where (c_{ij}) is the matrix exactly as in the proof of the Projection Theorem. Arguing exactly as in the proof of the Projection Theorem, we get

$$F(V) = \bigcup_i F(Z(g_i)).$$

This proves that that $F(V)$ is a subvariety of Ω_2, whenever V is globally defined in Ω as an intersection of zero-varieties. For the general case, pick $w_0 \in F(V)$, and choose $D_1, \ldots, D_k,\ N$ as in checkpoint, where the D_i's are chosen so that the preceding special case can be applied to show that $F(V \cap D_i)$ is a subvariety of N, for each i. Since

$$N \cap F(V) = \bigcup_{i=1}^{k} F(V \cap D_i),$$

we are done.

$\square$

CHAPTER 4

Alexander's Theorem

We have shown earlier that there is no direct generalization of the Riemann mapping theorem in $\mathbb{C}^n$, $n > 1$ (Theorem 1.7). We also proved that the automorphism group of the ball is transitive (Theorem 1.5). It turns out that transitivity is very special and in fact characterizes the ball uniquely, up to biholomorphism, among bounded smooth domains. In this chapter, we exploit the transitivity of the ball to prove a fascinating theorem, due to Alexander. The theorem states that for $n > 1$, any proper holomorphic map from the unit ball into itself is automatically an automorphism!

THEOREM 4.1 (Alexander [**1**]). *If $n > 1$, and F a proper holomorphic map of $\mathbb{B}$ into $\mathbb{B}$, then $F \in Aut(\mathbb{B})$.*

As the proof of Alexander's theorem requires a lot of work, the proof is presented in the final section. In the first section, we summarize some simple consequences of the classical Schwarz's lemma for the unit disk in $\mathbb{C}$. In the next section, we use these results to prove the main results needed for the proof of Alexander's theorem.

4.1. Consequences of Schwarz's lemma

We now state some simple consequences of Schwarz' lemma. The first one involves balanced sets, i.e., sets $E \subseteq \mathbb{C}^n$ in which $\lambda z \in E$, $|\lambda| \leq 1$, when $z \in E$. For the proofs, check Chapter 8 and 15 of [**7**].

THEOREM 4.2. *Suppose that :*

(i) *Ω_1 and Ω_2 are balanced regions in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively,*
(ii) *Ω_2 is convex and bounded,*
(iii) *$F : \Omega_1 \to \Omega_2$ is holomorphic.*

Then

(i) *$F'(0)$ maps Ω_1 into Ω_2, and*
(ii) *$F(r\Omega_1) \subseteq r\Omega_2$ $(0 < r \leq 1)$ if $F(0) = 0$.*

THEOREM 4.3. *If $F : \mathbb{B}_n \to \mathbb{B}_n$ is holomorphic and $F'(0)$ is an isometry of $\mathbb{C}^n$, then $F(z) = F'(0)z$, $\forall z \in \mathbb{B}_n$.*

THEOREM 4.4. *If $F : \mathbb{B}_n \to \mathbb{B}_m$ is holomorphic, $a \in \mathbb{B}_n$, and $F(a) = b$, then*

$$|\phi_b(F(z))| \leq |\phi_a(z)| \ (z \in \mathbb{B}_n).$$

Equivalently,

$$\frac{|1 - \langle F(z), F(a)\rangle|^2}{(1-|F(z)|^2)(1-|F(a)|^2)} \leq \frac{|1-\langle z, a\rangle|^2}{(1-|z|^2)(1-|a|^2)}.$$

Denote

$$D_z := \{\lambda z : \lambda \in \mathbb{C},\ |\lambda z| < 1\}.$$

THEOREM 4.5. *Suppose :*

(i) *Ω_1 and Ω_2 are regions in $\mathbb{B}$, $0 \in \Omega_1 \cap \Omega_2$,*
(ii) *F is a biholomorphic map of Ω_1 into Ω_2,*
(iii) *some point $p \in \Omega_1$, $p \neq 0$, has a neighbourhood $N_p \subseteq \Omega_1$ with the property that $D_z \subseteq \Omega_1$ and $D_{F(z)} \subseteq \Omega_2$ when $z \in N_p$.*

Then, F extends to an unitary operator on $\mathbb{C}^n$.

LEMMA 4.6. *If $F : \mathbb{B} \to \mathbb{B}$ is holomorphic, $F(0) = 0$, and $|(J_{\mathbb{C}}F)(0)| = 1$, then F is unitary.*

LEMMA 4.7. *Let $S := \partial\mathbb{B}$ and assume $n > 1$. If $0 < t < r < 1$, $\zeta \in S$, $a = r\zeta$, Ω is a region such that*

$$\{z \in \mathbb{B} : t < Re\,\langle z, \zeta\rangle\} \subseteq \Omega \subseteq B,$$

and $\delta = \frac{(1-r)(1+t)}{(1-rt)}$, then $\phi_a(\Omega)$ contains :

(i) *all $w \in \mathbb{B}$ with $|\langle w, \zeta\rangle| < 1 - \delta$, and*
(ii) *all D_η, where $\eta \in S$ and $|\langle \eta, \zeta\rangle| < 1 - \delta$.*

4.2. The main results

In this section, we present the two main results needed to prove Alexander's theorem. In the remainder of this report we will use S to denote $\partial\mathbb{B}$.

THEOREM 4.8. *Let $n > 1$. Suppose, for $i = 1, 2$, that Ω_i is a sub-domain of $\mathbb{B}$ whose boundary $\partial\Omega_i$ contain an open set Γ_i of S, and that F is a biholomorphic map of Ω_1 onto Ω_2. If there is a sequence $\{a_k\}$ in Ω_1, converging to a point $\alpha \in \Gamma_1$ which is not a limit point of $\mathbb{B} \cap \partial\Omega_1$, such that the points $b_k := F(a_k)$ converge to a point $\beta \in \Gamma_2$ which is not a limit point of $\mathbb{B} \cap \partial\Omega_2$, then F extends to an automorphism of $\mathbb{B}$.*

PROOF. Let a'_k and b'_k be such that $a_k = a'_k|a_k|$ and $b_k = b'_k|b_k|$. We can choose, $0 < t < 1$ such that, for large k, Ω_1 contains all $z \in \mathbb{B}$ with $t < Re\langle z, a'_k\rangle$, and Ω_2 contains all $w \in \mathbb{B}$ with $t < Re\langle w, b'_k\rangle$; also $t < |a_k|, t < |b_k|$.

Define $G_k := \phi_{b_k} \circ F \circ \phi_{a_k}$. Then, G_k is a biholomorphic map of $\Omega_1^k := \phi_{a_k}(\Omega_1)$ onto $\Omega_2^k := \phi_{b_k}(\Omega_2)$, with $G_k(0) = 0$. By Lemma 4.7, Ω_1 and Ω_2 contain a ball $(1-\delta_k)\mathbb{B}$, where $\delta_k \to 0$ as $k \to \infty$. Preceding G_k and G_k^{-1} by a scaling, and applying Theorem 4.2, we get

$$|J_{\mathbb{C}} G_k(0)| \leq (1-\delta_k)^{-n}, \;\; J_{\mathbb{C}} G_k^{-1}(0)| \leq (1-\delta_k)^{-n}.$$

Hence, $1-\delta_k \leq |J_{\mathbb{C}}(0)|^{\frac{1}{n}} \leq (1-\delta_k)^{-1}$. On a fixed Ω_1^k, the sequence $\{G_\nu\}_{\nu \geq k}$ converges, uniformly on compact subsets of Ω_1^k to a holomorphic map G. By Lemma 4.6, G extends to an unitary operator U on $\mathbb{B}$. Fix $0 < c < \frac{1}{100}$. For each k, let Y_k be the set of all z, $0 < |z| < 1-c$ such that $D_z \subseteq \Omega_1^k$ and $D_{Uz} \subseteq \Omega_2^k$. By Lemma 4.7, Y_k increases to $\mathbb{B}$. Therefore, we can choose a k, fixed from now, such that

$$|G_k(z) - Uz| < c \;\text{ if } |z| \leq 1-c,$$

and Y_k contains an open ball of radius $2c$. Let p be its centre. If $|z-p| < c$ and $w = G_k(z)$, then $D_z \subseteq \Omega_1^k$, and since

$$|U^{-1}w - p| = |w - Up| = |G_k(z) - Uz + Uz - Up| < 2c,$$

We have $U^{-1}w \in Y_k$, hence $D_w \subseteq \Omega_2^k$. Applying theorem 4.5 to G_k, we get that G_k is unitary and as $F = \phi_{b_k} \circ G_k \circ \phi_{a_k}$, we have $F \in Aut(\mathbb{B})$. □

For $\alpha > 1$ and $\xi \in S$, let $\Gamma_\alpha(\xi)$ denote the set of points in $\mathbb{C}^n$ such that

$$|1 - \langle z, \xi \rangle| < \frac{\alpha}{2}(1-|z|^2).$$

Clearly, $\Gamma_\alpha(\xi) \subseteq \mathbb{B}$.

DEFINITION 4.9. A function $F : \mathbb{B} \to \mathbb{C}$ is said to have *admissible limit* or K-limit λ at $\xi \in S$, if for ever $\alpha > 1$, and every sequence $\{z_i\} \subseteq \Gamma_\alpha(\xi)$ that converges to ξ, $F(z_i) \to \lambda$.

We now state a theorem of Koranyi and Vági, on the existence of admissable limits. Their result is applicable to a much more general setting. However, even for $\mathbb{B}$, the proof relies on ideas quite divergent from those discussed here. Hence, we direct the reader to a proof given in Chapter 5 of [**7**].

THEOREM 4.10 (Koranyi-Vági Theorem). *Let $f \in \mathcal{O}(\mathbb{B})$ be a non-zero holomorphic function. Then for almost every (with respect to the Lebesgue Measure on S) $\xi \in S$, the admissible limit of f exists, and is non-zero.*

LEMMA 4.11 (Henkin [**4**]). *Suppose that $F : \mathbb{B} \to \mathbb{B}$ is proper holomorphic map, with $F(0) = 0$. Then F has a continuous extension to the closure of $\mathbb{B}$, and there is a constant $A < \infty$ such that*

$$F(\Gamma_\alpha(\xi)) \subseteq \Gamma_{A\alpha}(F(\xi)),$$

for every $\xi \in S$ and every $\alpha > 1$.

PROOF. For $w \in \mathbb{B}$, let $\{p_i(w)\}$ be the inverse images of w under F. Define

$$\mu(w) = \max_i |p_i(w)|^2 \quad (w \in \mathbb{B}).$$

Then $\mu \in C(\mathbb{B}),\ \mu < 1$. As μ is plurisubharmonic outside the critical set of F and the critical set is a zero-variety (Theorem 3.13), we conclude, by Riemann's removable singularities theorem for plurisubharmonic functions, that μ is plurisubharmonic in $\mathbb{B}$. Since F is proper, there exists $c < 1$ such that $|z| \le c$ whenever $|F(z)| \le \frac{1}{2}$. Thus, $\mu(w) \le c^2$ if $|w| \le 1/2$ and therefore

$$|p_i(w)|^2 \le \mu(w) \le 1 - c_1(1 - |w|^2) \quad (w \in \mathbb{B})),$$

where $c_1 > 0$ is chosen so that the right side is c^2 when $|w| = \frac{1}{2}$. Hence, there is a constant $A < \infty$, such that

$$1 - |F(z)|^2 \le A(1 - |z|^2) \quad (z \in \mathbb{B}). \tag{4.1}$$

From Theorem 4.4, F satisfies

$$|1 - \langle F(z), F(a)\rangle \le A|1 - \langle z, a\rangle|,$$

for all $z, a \in \mathbb{B}$. When z and a tend to the same boundary point, the right side tends to zero, and hence so does the left. Hence, F extends continuously to $\mathbb{B}$. Since $F(0) = 0$, by Theorem 4.2, we get $|F(z)|^2 \le |z|^2$ and hence

$$\frac{1}{1 - |F(z)|^2} \le \frac{1}{1 - |z|^2}.$$

Multiplying with (4.1) gives

$$\frac{|1 - \langle F(z), F(\xi)\rangle|}{1 - |F(z)|^2} \le A\frac{|1 - \langle z, \xi\rangle}{1 - |z|^2},$$

for $z \in \mathbb{B},\ \xi \in S$. Hence, $F(\Gamma_\alpha(\xi)) \subseteq \Gamma_{A\alpha}(F(\xi))$. □

4.3. Proof of Alexander's theorem

We now give the proof of Alexander's theorem.

PROOF. We may assume that $F(0) = 0$. Let the multiplicity of F be m, and let $p_1(w), \ldots, p_m(w)$ be as before. F extends continuously to $\overline{\mathbb{B}}$, hence we can extend the definition of $\#(w)$ to points in S. The first part of the proof is to prove that there is a point $w \in S$ such that $\#(w) = m$.

Step 1. $\#(w) \ge m$ *for almost all* $\eta \in S$.

Let Λ be a linear transformation on $\mathbb{C}^n$ that separates the points $p_i(w_0)$ for some regular value $w_0 \in \mathbb{B}$, i.e., $\Lambda p_i(w_0) \neq \Lambda p_j(w_0),\ i \neq j$. By Riemann's removable singularities theorem, there is an $h \in H^\infty(\mathbb{B})$ such that

$$h(w) = \prod_{i<j} \left[\Lambda p_i(w) - \Lambda p_j(w)\right]^2,$$

for every regular value of F, and by our choice of Λ, $h \not\equiv 0$. By Theorem 4.10, for almost every $\eta \in S$, the admissible limit of h exists and is non-zero. Let η_0 be one such point. There is an approach region $\Gamma_\alpha(\eta)$ and a $\delta > 0$ such that $|h(w)| > \delta$ on $\Gamma_\alpha(\eta)$. Therefore, every point in $\Gamma_\alpha(\eta)$ is a regular value of F, and hence by the definition of h, there exists $\varepsilon > 0$ and functions $p_i \in \mathcal{O}(\Gamma_\alpha(\eta))$, which invert F, and such that

$$|p_i(w) - p_j(w)| > \varepsilon \;\; (w \in \Gamma_\alpha(\eta)\; i \neq j).$$

Let $\{w_k\}$ be a sequence in $\Gamma_\alpha(\eta)$ that converges to η. We may assume that $\lim_{k\to\infty} p_i(w_k)$ exists for $1 \le i \le m$. Therefore, we get m distinct points, $\xi_i := \lim_{k\to\infty} p_i(w_k)$ on S such that $F(\xi_i) = \eta$. Thus, $\#(w) \geq m$.

Step 3. $\#(w) \le m$ *for almost every* $\eta \in S$.

We can find a countable class of linear functionals Φ, on $\mathbb{C}^n$, such that every finite set of points in $\mathbb{C}^n$ is separated by some $\Lambda \in \Phi$. Define

$$Q_\Lambda(t, w) := \prod_{i=1}^{m} (t - \Lambda p_i(w)), \tag{4.2}$$

for $\Lambda \in \Phi$, $t \in \mathbb{C}$, and w a regular value of F. The coefficients $g_{k,\Lambda}(w)$ in the expansion

$$Q_\Lambda(t, w) = t^m + \sum_{k=0}^{m-1} g_{k,\Lambda}(w) t^k, \tag{4.3}$$

are polynomial in $\Lambda p_i(w)$ and hence, extend to functions in $H^\infty(\mathbb{B})$ by Riemann's removable singularities theorem. For almost every $\eta \in S$, the admissible limit of every $g_{k,\Lambda}$ exists. Let η be one such point. Choose $\xi \in S$ such that $F(\xi) = \eta$. As $r \nearrow 1$, by Lemma 4.11, $F(r\xi)$ tends to η within some region $\Gamma_\alpha(\eta)$, so that

$$\lim_{r\nearrow 1} g_{k,\Lambda}(F(r\xi)) = g_{k,\Lambda}(\eta),$$

exists for all k and all Λ. Since $r\xi = p_i(F(r\xi))$ for some i, (4.2) shows that

$$Q_\Lambda(\Lambda r\xi, F(r\xi)) = 0,$$

and by (4.3), we get

$$(r\Lambda\xi)^m + \sum_{k=0}^{m-1} g_{k,\Lambda}(F(r\xi))(r\Lambda\xi)^k = 0.$$

Letting, $r \nearrow 1$, we see that $\Lambda\xi$ is a root of the polynomial of degree m, $Q_\Lambda(\cdot, \eta)$. If $\#(\eta) > m$, then we can choose $\Lambda \in \Phi$, which separate the $m + 1$ pre-images, which implies that $Q_\Lambda(\cdot, \eta)$ would have $m + 1$ roots, which is a contradiction.

Step 3. We set things up for an application of Theorem 4.8. Let $\eta \in S$, such that $F^{-1}(\eta) = \{\xi_1, \ldots, \xi_m\}$. Choose $r > 0$, such that $|\xi_i - \xi_j| > 3r,\ i \neq j$. Let β_i be the set of points in $z \in \mathbb{B}$ with $|z - \xi_i| < r$. $F(\overline{\mathbb{B} \cap \partial\beta_i})$ does not contain η for $1 \leq i \leq m$. Therefore, there exists $\delta > 0$ such that set

$$V = \{w \in \mathbb{B} : |w - \eta| < \delta\},$$

does not intersect any of the sets $F(\overline{\mathbb{B} \cap \partial\beta_i})$. Since F is an open mapping, each $F(\beta_i)$ is an open subset of $\mathbb{B}$, which contains no boundary point of $F(\beta_i)$. Clearly, $F(\beta_i)$ intersects V and hence by a connectedness argument, we get that $V \subseteq F(\beta_i)$. Set

$$\Omega_i = \beta_i \cap F^{-1}(V)\ \ (1 \leq i \leq m).$$

Then $F(\Omega_i) = V$. The sets β_i are pairwise disjoint and no point of $\mathbb{B}$ has more than m pre-images. Therefore, F is injective on each Ω_i and hence by Osgood's theorem, F is a biholomorphic map of Ω_i onto V. We have to check that there is an $\varepsilon > 0$ such that Ω_i contains all $z \in \mathbb{B}$ with $|z - \xi_i| < \varepsilon$. If this were false, there would be a sequence $\{z_k\}$ in $\beta_i \setminus \Omega_i$, converging to ξ_i. Since F is continuous on $\overline{\mathbb{B}}$, and $f(\xi_i) = \eta$, we have $F(z_k) \in V$, for large k. Therefore, for large $k, z_k \in \Omega_i$. By Theorem 4.8, $F \in Aut(\mathbb{B})$.

□

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