MA5371 – Assignment 4
Oct 13, 2016

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1. Which of the following subsets of $\mathbb{R}$ have (length) volume zero.
   a) $\mathbb{Q} \cap [0,1]$.
   b) $\mathbb{Q}^c \cap [0,1]$
   c) The Cantor set.

2. Let $X \subset \mathbb{R}^n$ be a bounded subset which is contained in a proper vector subspace of $\mathbb{R}^n$. Prove that X has volume zero.

3. Let $X \subset \mathbb{R}^n$ be a set of volume zero and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Prove that $T(X)$ has volume zero.

4. Let $\Omega \subset \mathbb{R}^n$ be a connected region and let $f : \Omega \to \mathbb{R}$ be a continuous function. Prove that there is a point $c \in \Omega$ such that
$$\int_\Omega f = f(c)\mathsf{vol}(\Omega).$$

5. Find the volume of the region in $\mathbb{R}^3$ bounded by the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.

6. Use Fubini's theorem to show that the mixed partial derivatives of $\mathcal{C}^2$-smooth function are equal.

7. (**Symmetry principle**) Let $R \subset \mathbb{R}^n$ be a rectangle that is symmetric about the $x_1 = 0$ hyperplane, i.e., $(x_1, \ldots, x_n) \in R \implies (-x_1, x_2, \ldots, x_n) \in R$. Suppose $f : R \to \mathbb{R}$ is an integrable function with the property that $f(x_1, \ldots, x_n) = f(-x_1, x_2, \ldots, x_n)$. Compute $\int_R f$. What if R were symmetric about the origin and f is an odd function? What about more general regions?

8. Let $B \subset \mathbb{R}^3$ be the open unit-ball. Evaluate
$$\int_B x^2.$$

9. Find the volume of the cone of radius r and height h using cylindrical coordinates and spherical coordinates.

10. Use the change of variables theorem to justify the use of polar, cylindrical and spherical coordinates.

    Note: Check the hypotheses of the change of variables theorem very carefully!