MA5371 – Assignment 2

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Throughout this assignment, unless otherwise specified, $U \subset \mathbb{R}^n$ will be an open set, $a \in U$ and $f : U \to \mathbb{R}^m$ will be a function.

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1. Let $X \subset \mathbb{R}^n$ be a non-compact set. Construct a function $f : X \to \mathbb{R}$ that is unbounded.

2. Let $a_1, \ldots, a_k \in \mathbb{R}^n$ be a fixed set of points. Show that the function

$$f(x) = \sum_{i=1}^{k} \|x - a_i\|^2$$

has a global minimum and find the minimum point.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be harmonic (i.e, f is $\mathcal{C}^2$-smooth and $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$). Assume $\frac{\partial^2 f}{\partial x^2}(a) \neq 0$. Prove that a cannot be an extremum of f.

4. Classify the critical points of the function $x^3 + e^{3y} - 3xe^y$.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be $\mathcal{C}^1$-smooth. Prove that f **cannot** be injective.

6. Given an example to show that the continuity of the partial derivatives is an essential hypothesis in the statement of the inverse function theorem.

7. Use the implicit function theorem to give a proof of the inverse function theorem.

8. Find the maximum and minimum of the function $x_1 + \cdots + x_n$ defined on $\mathbb{R}^n$ subject to the constraint $\|x\| = 1$.

9. Suppose $p, q > 0$ and $1/p + 1/q = 1$ and $x, y > 0$. Use Lagrange multipliers to prove that

$$\frac{x^p}{p} + \frac{y^q}{q} \geq xy.$$

10. Let $\mathcal{M}_{n\times n}$ denote the collection of all linear transformations from $\mathbb{R}^n$ to itself (you may think of them as square matrices, if you wish). Let $\|\cdot\|$ denote the operator norm (sup norm). Prove that:

    a) If $S, T \in \mathcal{M}_{n\times n}$, then
    $$\|S + T\| \leq \|S\| + \|T\|.$$
    Thus, we can a define a metric on $\mathcal{M}_{n\times n}$ by $d(S, T) := \|S - T\|$.

    b) Let $f : U \to \mathbb{R}^n$ be $\mathcal{C}^1$-smooth. Prove that the map $\Lambda : U \to \mathcal{M}_{n\times n}$ given by $x \mapsto Df(x)$ is a continuous mapping.

c) Let $GL_n(\mathbb{R})$ denote the subspace of invertible $n \times n$ matrices. Prove that the map $A \mapsto A^{-1}$ from $GL_n(\mathbb{R})$ to itself is a continuous mapping.

d) Let $A \in \mathcal{M}_{n\times n}$ satisfy $\|A - I\| \leq 1/2$. Prove that A is invertible.

11. Let $P : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree k. Let $a \in \mathbb{R}^n$. What is the Taylor polynomial of P of degree k at the point a?

12. Is every $\mathcal{C}^\infty$-smooth function automatically real-analytic?