MA5371 – Assignment 1

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Throughout this assignment, unless otherwise specified, $U \subset \mathbb{R}^n$ will be an open set, $a \in U$ and $f : U \to \mathbb{R}^m$ will be a function.

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1. Let f be differentiable at a. Prove that the derivative $Df(a)$ is unique.

2. If $f = (f_1, \dots, f_m)$, show that f is differentiable at a iff each f_i is differentiable at a.

3. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Show that for all $0 \neq v \in \mathbb{R}^n, a \in \mathbb{R}^n, D_v T(a)$ exists and calculate it. Is T differentiable? If so, what is its derivative?

4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function such that $\|f\| \leq \|x\|$. Show that f is differentiable at 0.

5. Let $U \subset \mathbb{R}^2$ be an open set $f : U \to \mathbb{R}$. We say that f is independent of the second variable if for each $(x_0, y_0) \in U$, the one variable function $f(x_0, y)$ defined on the set $(\{x_0\} \times \mathbb{R}) \cap U$ is constant.

   a) Show that if f is independent of the second variable, then $\frac{\partial f}{\partial y} \equiv 0$ on U.

   b) Let $U := \{(x, y) \in \mathbb{R}^2 : x < 0, \textbf{or } x \geq 0 \text{ and } y \neq 0\}$ and suppose $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} \equiv 0$ on U. Show that f must be constant.

   c) Construct an open set $U \subset \mathbb{R}^2$ and a function $f : U \to \mathbb{R}$ such that $\frac{\partial f}{\partial y} \equiv 0$ on U but yet f is **not** independent of the second variable.

6. Let $U \subset \mathbb{R}^n$ be open and convex (i.e., $\forall x, y \in U, tx + (1-t)y \in U, t \in [0, 1]$) and suppose $Df \equiv 0$ on U. Prove that f is constant.

7. Let $g : (a, b) \to \mathbb{R}^n$ be a twice-differentiable parametrized curve. Prove that g has constant speed iff the velocity and acceleration vectors are orthogonal for each $t \in (a, b)$.

8. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(0) = 0$ and $f(x, y) = xy\frac{x^2 - y^2}{x^2 + y^2}$ when $(x, y) \neq 0$. Show that the mixed partial derivatives of f are not equal at 0. Why does this happen?

9. Define $f : \mathbb{R} \to \mathbb{R}$ by $\exp(-x^{-2})$ when $x \neq 0$ and $f(0) = 0$. Show that $f \in \mathcal{C}^\infty(\mathbb{R})$. Compute **all** the derivatives of f at 0.

   Hint: Use L'Hospital's rule and induction.

10. Let $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^p$ be a bilinear function (refer to Wikipedia if you do not know the definition). Prove that

    a) $\lim_{(h,k) \to 0} \frac{\|f(h,k)\|}{\|x\|} = 0$.

    b) $Df(a, b)(x, y) = f(a, y) + f(x, b)$.