Solutions to MA5360 – Quiz 2

1. (6 Marks) Let $D \subset \mathbb{C}$ be a domain and let $f \in H(D)$. Suppose that $f = u + iv$ and there are constants $a, b \in \mathbb{R}, a \neq 0, b \neq 0$ such that $au + bv$ is constant on D. Prove that f is constant.

   This is a straightforward application of the CR equations.

2. (6 Marks) Show that the fractional linear transformations that are real on the real axis are precisely those that can be expressed in the form $(az + b)/(cz + d)$, where a, b, c and d are real.

   As fractional linear transformations must take circles on the Riemann sphere to circles on the Riemann sphere, the point ∞ must get mapped to ∞. This means that the transformation is of the form $\frac{az+b}{d}$. Setting $z = 0$, we see that b/d is real and it now follows easily that a/d is real and we are done.

3. (6 Marks) Write down the power series expansion of $\operatorname{Log}(1 - z)$ about 0. What is the radius of convergence of this series expansion?

   This is example 2.34 from the textbook.

4. (6 Marks) Let $f : D(a, R) \to \mathbb{C}$ be a bounded function and suppose that $f \in H(D(a, R) \setminus \{a\})$. Prove that f has an anti-derivative on $D(a, R)$.

   From the results proved in class, f has an anti-derivative iff the line integral of f on the boundary of any triangle is 0. If the triangle misses the point a then by Cauchy's theorem there is nothing to prove. On the other hand, if the triangle contains the point a, subdivide the triangle into smaller triangles with a being contained in one triangle. Orient the boundaries of the smaller triangles in such a way that the integral of f on the boundary of the large triangle is nothing but the sums of the integrals on the boundaries of the smaller triangles. There will be many cancellations and (as seen in class) the only term that remains will be the integral around the boundary of the triangle containing a. Shrinking this triangle and using the fact that f is bounded delivers the result.

5. (6 Marks) Let f be a holomorphic function on the open set U and such that $|f(z) - 1| < 1 \; \forall z \in U$. Let γ be a closed path contained in U. Prove that
   $$\int_\gamma \frac{f'(z)}{f(z)} dz = 0.$$

   Hint: You will have to use one of the previous problems. Figure out which one and use it (even if you have not solved it).

   Integral around any closed path is zero iff the function has an anti-derivative. f'/f has an anti-derivative iff f has a continuous branch of the logarithm. Problem 3 now delivers the result.