Solutions to MA5360 – Quiz 1

1. (5 Marks) Let $D \subset \mathbb{C}$ be a domain and let $f \in H(D)$. Suppose $\arg f(z)$ is a constant modulo $2\pi\mathbb{Z}$. What can you say about f?

   The function f must b constant. The idea behind the proof is very simple. The image of the function f all lie on a single ray. We just rotate the ray to make it the real-axis. Fix $z_0 \in D$ and note that the function $g(z) := e^{-i \arg f(z_0)} f(z)$ has the property that $\arg g(z) = 0$ modulo $2\pi\mathbb{Z}$. This means that g is a real-valued function and hence must be a constant function by a result proved in class. Hence, $f(z)$ is also a constant function.

2. (10 Marks) Let $\gamma_n : [0, 1] \to \mathbb{C}$ be a sequence of continuous curves that converges uniformly to the curve $\gamma : [0, 1] \to \mathbb{C}$. Let $z \in \mathbb{C} \setminus \gamma^*$. Show that $\operatorname{Ind}(\gamma, z) = \operatorname{Ind}(\gamma_n, z)$ for n suitably large.

   First of all, $\operatorname{Ind}(\gamma_n, z)$ is well-defined for large n. This is because $z \notin \gamma^*$ and hence $z \notin \gamma_n^*$ for n suitably large as $\gamma_n \to \gamma$ uniformly. Furthermore, $|\gamma_n(t) - \gamma(t)| < \varepsilon$ for n suitably large. Now the proof is exactly same as that for proving that the winding number is a continuous function.

3. (5 Marks) Using the Cauchy–Riemann equations, determine whether the following functions are entire:

   a) $|z|^2$.
   b) z^2.
   c) e^z.

   This is straightforward. First note that all the three functions above are $\mathbb{R}$-differentiable. In fact, all these functions are $\mathcal{C}^\infty$-smooth on the whole of $\mathbb{C}$. So the only thing to check is which of these functions satisfy the CR equations. The first one satisfies the CR equations only at 0 and is therefore not entire. The other two satisfy the CR equations on the whole of $\mathbb{C}$ and are therefore entire.