Proof that the winding number is a continuous function

Let $\gamma : [a, b] \to \mathbb{C}$ be a continuous curve. Let $z, w \in \mathbb{C} \setminus \gamma^*$. We will show that if w is sufficiently close to z, then $\text{Ind}(\gamma, z) = \text{Ind}(\gamma, w)$. Let

$$(\gamma - z)(t) = r_1(t)e^{i\phi_1(t)}$$
$$(\gamma - w)(t) = r_2(t)e^{i\phi_2(t)}.$$

Fix $\varepsilon > 0$ very small. Then we can choose w so close to z so that $|\phi_1(t) - \phi_2(t)| < \varepsilon \mod 2\pi\mathbb{Z}$. Furthermore, we can assume that $|\phi_1(a) - \phi_2(a)| < \varepsilon$. We claim that $|\phi_1(t) - \phi_2(t)| < \varepsilon \ \forall t \in [a, b]$. Clearly, the set of points that satisfy $|\phi_1(t) - \phi_2(t)| < \varepsilon$ is an open subset of $[a, b]$ and clearly the set of points that satisfy $|\phi_1(t) - \phi_2(t)| > \varepsilon$ is also open. The set of points that satisfy $|\phi_1(t) - \phi_2(t)| = \varepsilon$ is empty. By connectedness of $[a, b]$, our claim is proved. From the claim, it follows that $|\text{Ind}(\gamma, z) - \text{Ind}(\gamma, w)| < 2\varepsilon$ and hence $\text{Ind}(\gamma, z) = \text{Ind}(\gamma, w)$ for w suitably close to z.