Outline of solution to assignment 1 question 6

Fill in the details of the steps given below to complete the proof. If someone submits a different (correct) proof, they get bonus points.

1. Without loss of generality, assume that U is connected. Let $g : U \to \mathbb{C}$ be a continuous branch of the square root function. Then we have
$$g(z)^2 = z.$$
Use this to show that $g \in H(U)$ and that $g'(z) = \frac{1}{2g(z)}$. You might freely use the chain rule, product rule, etc. Your are just proving the familiar fact that $\frac{\partial \sqrt{x}}{\partial x} = \frac{1}{2\sqrt{x}}$ so don't worry too much about being rigorous.
2. In this step, we will show that if $\gamma : [0, 1] \to U$ is any closed path then $\mathrm{Ind}(\gamma, 0) = 0$. Suppose not, then
$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z} = k \in \mathbb{Z}, k \neq 0.$$
Note that $g(\gamma(z))$ is also a closed path. Hence,
$$\mathrm{Ind}(g(\gamma), 0) = \frac{1}{2\pi i}\int_{g(\gamma)} \frac{dz}{z} = \frac{1}{2\pi i}\int_0^1 \frac{g'(\gamma(t))\gamma'(t)}{g(\gamma(t))} dt = \frac{1}{2\pi i}\int_0^1 \frac{\gamma'(t)}{2\gamma(t)dt} = k/2,$$
which shows that the winding number of γ with respect to 0 is an even number. But any closed curve in the plane with non-zero winding number around the origin contains in its image a simple closed curve with winding number one around the origin, and this leads to a contradiction.
3. Fix $z_0 \in U$ and define $f(z_0)$ to be some arbitrary value in $\arg z_0$. For any z in U, let $\gamma : [0, 1] \to U$ be a path from z_0 to z and let ϕ be the branch of $\arg$ along γ that satisfies $\phi(z_0) = f(z_0)$. Set $f(z) = \phi(z)$. Prove that f is well-defined and that it is the required continuous branch of $\arg z$ on U.