CONSTRUCTING MÖBIUS TRANSFORMATIONS WITH SPHERES

Rob Siliciano[a]

[a]Princeton University

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Abstract. Every Möbius transformation can be constructed by stereographic projection of the complex plane onto a sphere, followed by a rigid motion of the sphere and projection back onto the plane, illustrated in the video *Möbius Transformations Revealed*. In this article we show that, for a given Möbius transformation and sphere, this representation is unique.

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1   Introduction

In November of 2007 a curious addition appeared on the homepage of YouTube, the internet video sharing site. Alongside the more standard fare of talking cats and a video explaining how to charge an iPod with a potato, the "Featured Video" list included a short film about high-level mathematics. The video, called *Möbius Transformations Revealed*, was created by Douglas Arnold and Jonathan Rogness, of the University of Minnesota, who were as surprised as anybody when it went viral. The video is comprised of visual representations of Möbius transformations, such as the image in Figure 1, with a main goal of demonstrating the beauty of mathematics to viewers. However, the film does in fact illustrate a specific theorem which states a given Möbius transformation can be constructed using a sphere, stereographic projection and rigid motions of the sphere, as described by Arnold and Rogness in [2] and detailed below.

The purpose of this article is to answer an open question from that article, namely: given a specific Möbius transformation, *in how many different ways* can the transformation be constructed using a sphere? The main result shows that for any given Möbius Transformation and so-called *admissible* sphere there is exactly one rigid motion of the sphere with which the transformation can be constructed.

Before continuing, readers may wish to browse to [1] to view *Möbius Transformations Revealed*.
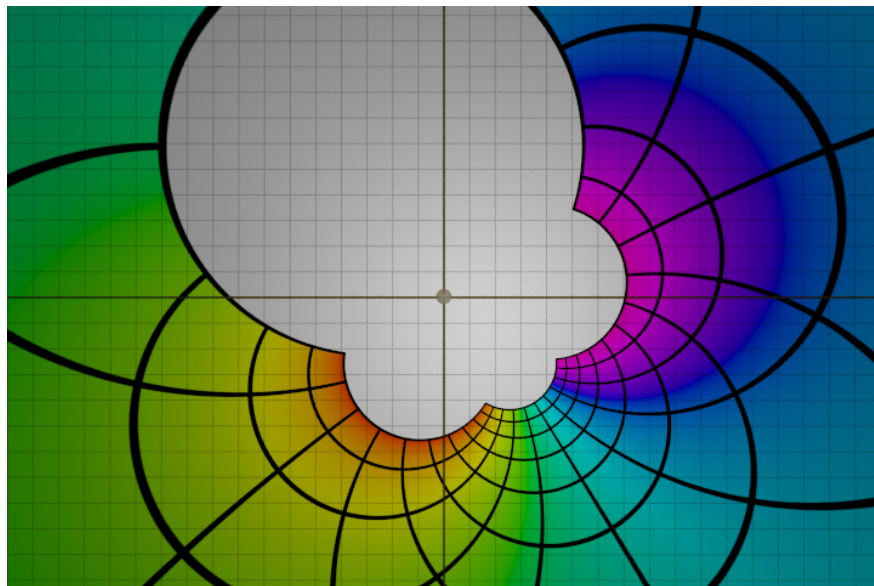

Figure 1: Visual representation of a Möbius transformation

2   Background

This section gives basic definitions for those readers who are unfamiliar with Möbius transformations and stereographic projection. Full details and proofs can be found in standard complex analysis texts, such as [3].

2.1   Möbius Transformations

A *Möbius transformation*, also known as a *linear fractional transformation*, is a function $f : \mathbb{C} \to \mathbb{C}$ of the form

$$f(z) = \frac{az + b}{cz + d},$$

for $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

These functions can be extended to the Riemann Sphere $\mathbb{C}_\infty$, the complex numbers with the "infinity point" ∞. In $\mathbb{C}_\infty$, $f(-d/c)$ has a value, since fractions with a zero denominator exist, and are ∞. On the Riemann Sphere, Möbius transformations are bijective, and their inverses are also Möbius transformations.

An important property of these functions is that a Möbius transformation is uniquely determined by the effect of any three distinct points in $\mathbb{C}_\infty$. So, for distinct $z_1, z_2, z_3 \in \mathbb{C}_\infty$ and distinct $w_1, w_2, w_3 \in \mathbb{C}_\infty$, there exists a unique Möbius transformation f for which $z_i \mapsto w_i$. In particular, the function

$$f(z) = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

is the unique Möbius transformation which sends z_1, z_2, and z_3 to 0, 1, and ∞, respectively.

2.2   Stereographic Projections

Using stereographic projection, the extended complex plane can be mapped bijectively onto certain spheres in $\mathbb{R}^3$. We will identify $\mathbb{R}^3$ with $\mathbb{C} \times \mathbb{R}$. In this identification, we will use ordered pairs rather than ordered triples for points in $\mathbb{R}^3$, so that a point in the complex plane may be referred to as either α or $(\alpha, 0)$ as needed. Following [2], we will call a sphere $S \in \mathbb{R}^3$ *admissible* if it has radius 1 and is centered at $(\alpha, c) \in \mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3$ with $c > -1$. Geometrically this means S is a unit sphere whose "north pole" is above the complex plane.

Definition 1. *Given an admissible sphere S centered at $(\gamma, c) \in \mathbb{R}^3$, the* Stereographic Projection *from S to $\mathbb{C}$ is the function $P_S : S \to \mathbb{C}_\infty$ which maps the top of S, $(\gamma, c + 1)$, to ∞, and maps any other point on the sphere to the intersection of the complex plane with the line extending from $(\gamma, c + 1)$ through the point.*

For a sphere S above the complex plane, P_S acts similar to an overhead projector, where light goes through a transparent sheet onto a screen, projecting the colors of the sheet to

the screen. In stereographic projection, the light source is placed at the top of a transparent sphere S, and the complex plane is the screen. Figure 2 shows two examples.

Given $P_S : S \to \mathbb{C}_\infty$, the inverse function $P_S^{-1} : \mathbb{C}_\infty \to S$ is a bijective mapping of the extended plane onto the sphere; specifically, P_S^{-1} is the function that sends ∞ to $(\gamma, c+1)$ and $\alpha \in \mathbb{C}$ to the nontrivial intersection of S and the line between $(\alpha, 0)$ and $(\gamma, c+1)$. Mathematicians sometimes refer to both P_S and P_S^{-1} as stereographic projections, leaving the reader to determine from context whether the function in question is a projection or its inverse.

2.3   Rigid Motions

The final functions that will be used in our construction of of Möbius transformations are *rigid motions*. These maps are called "rigid" because they correspond to the ways one can move a rigid, physical object without breaking or distorting it.

Definition 2. *A* rigid motion *of $\mathbb{R}^3$ is an isometry $\mathbb{R}^3 \to \mathbb{R}^3$ that preserves orientation. When using an admissible sphere S, we will call a rigid motion T admissible if the sphere $T(S)$ is also admissible.*

Rigid motions are functions made of the composition of translations and rotations about lines in $\mathbb{R}^3$, sending shapes to congruent shapes without reflection. In particular, lines are sent to lines, circles are sent to circles, and spheres are sent to spheres.

3   Modeling Möbius Transformations with Spheres

In [2], Arnold and Rogness proved the following result relating Möbius transformations, stereographic projections, and rigid motions:
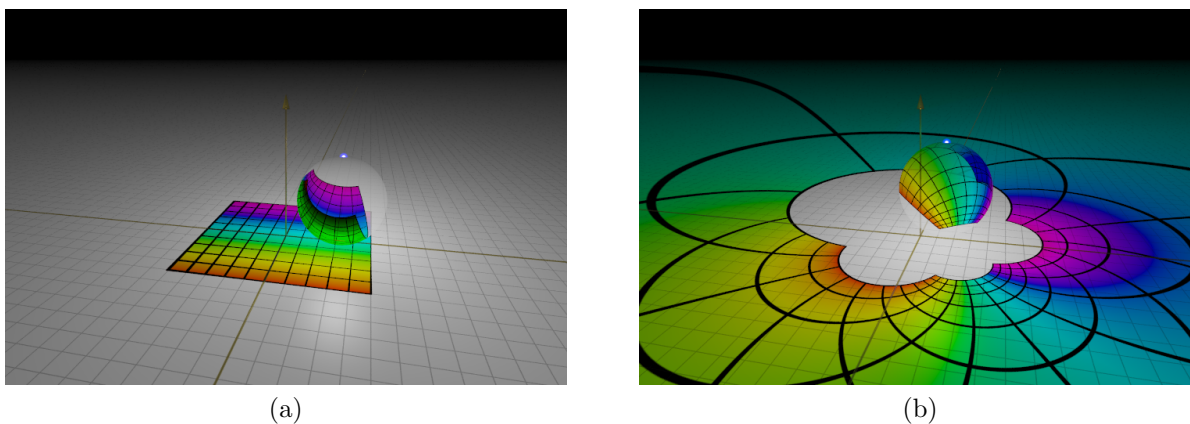

(a)   (b)

Figure 2: Stereographic projections from spheres to the complex plane

(a)   (b)
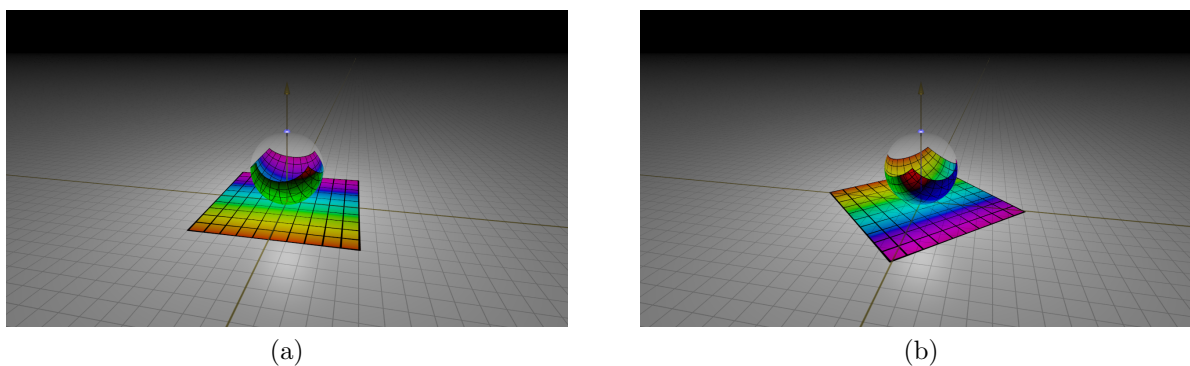
Figure 3: The Möbius transformation $f(z) = \frac{1-i}{\sqrt{2}} z$

Theorem 3. *A complex mapping is a Möbius transformation if and only if it can be obtained by steregraphic projection of the complex plane onto an admissible sphere in $\mathbb{R}^3$, followed by a rigid motion of the sphere in $\mathbb{R}^3$ which maps it to another admissible sphere, followed by stereographic projection back to the plane.*

In other words, given any admissible sphere S, and any admissible rigid motion T, the function $f = P_{T(S)} \circ T \circ P_S^{-1}$ is a Möbius transformation. Furthermore, given any Möbius transformation $f(z)$ there exists an admissible sphere S, and an admissible rigid motion T such that $f = P_{T(S)} \circ T \circ P_S^{-1}$.

We can observe that the sphere and rigid motion corresponding to a given Möbius transformation are not necessarily unique. A simple example is the identity Möbius transformation $I(z) = z$. For any admissible sphere, the identity rigid motion I gives $P_{I(S)} \circ I \circ P_S^{-1} = f$.

A more elaborate example is illustrated in Figures 3 and 4, which show two different ways of construction the Möbius transformation which rotates the complex plane by an angle of $5\pi/4$ counterclockwise about the origin. In each figure, the first picture shows the points on the plane copied onto the initial sphere; the second shows the result after the rigid motion and projection back onto the plane. In Figure 3 the rigid motion is a rotation about the vertical axis of the sphere, whereas the sphere in Figure 4 actually moves to a different location.

Although Theorem 3 states that any Möbius transformation can be modeled with a sphere, it stops short of classifying the many ways this can happen with various spheres and rigid motions. This gap is filled by the following theorem, the main result of this paper.

Theorem 4. *Let f be a Möbius transformation. For any admissible sphere S, there exists a unique rigid motion T such that $P_{T(S)} \circ T \circ P_S^{-1} = f$.*

To prove uniqueness, we will show that any two admissible rigid motions that model the Möbius transformation for a given sphere must be the same rigid motion. To prove existence, we will construct the necessary rigid motion.
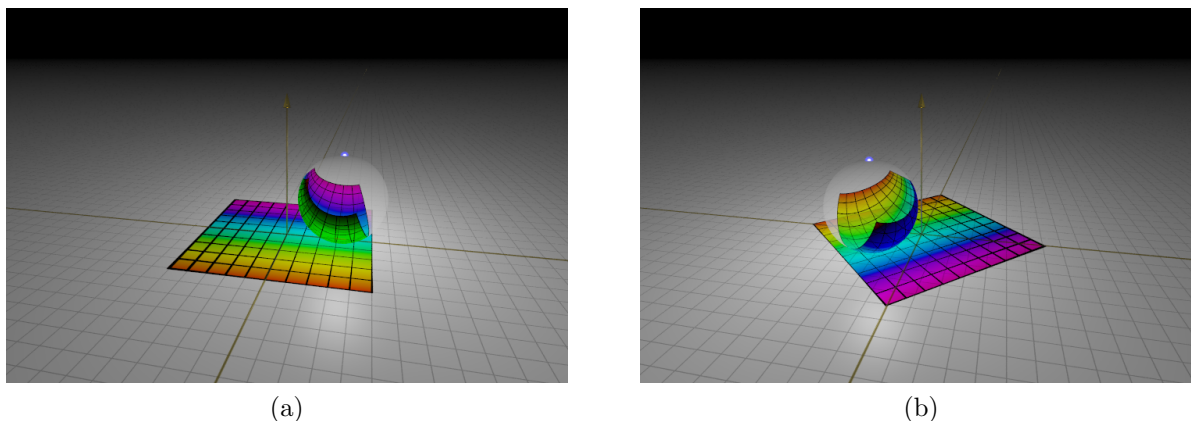
(a)   (b)

Figure 4: An alternate construction of $f(z) = \frac{1-i}{\sqrt{2}}z$

3.1   Uniqueness

We will now prove that any construction of a Möbius transformation by a sphere and rigid motion is unique up to the sphere. To do this, we will show that, given an admissible sphere, any two rigid motions that form the same Möbius transformation must be equal. We begin with a technical lemma.

Lemma 5. *Given admissible sphere S and admissible rigid motion T, if $id = P_{T(S)} \circ T \circ P_S^{-1}$, then $T = I$.*

Proof. The proof presented here uses elementary geometry and proceeds in three steps. First we show that T must map the north and south poles of S onto the north and south poles, respectively, of $T(S)$. Next we demonstrate that the "north-south axes" of S and $T(S)$ are on the same vertical line above a point in $\mathbb{C}$; this means the only valid choice for the rigid motion T would be a vertical translation followed by a rotation of S onto itself about its north-south axis. Finally, we use the properties of T to show that the translation and rotation are trivial—that is to say, $T = id$.

Beginning with $id = P_{T(S)} \circ T \circ P_S^{-1}$, we compose on the left with $P_{T(S)}^{-1}$ or on the right with P_S to obtain

$$P_{T(S)}^{-1} = T \circ P_S^{-1} \tag{1}$$

$$P_S = P_{T(S)} \circ T \tag{2}$$

respectively.

From (1) we have $P_{T(S)}^{-1}(\infty) = T\left(P_S^{-1}(\infty)\right)$, implying that T maps the top of sphere S to the top of sphere $T(S)$. Furthermore, because T is rigid, it maps antipodal points on S to antipodal points on $T(S)$. Hence, T maps the bottom of S to the bottom of $T(S)$.

If S is centered at (γ, c) then $P_S(\gamma, c - 1) = (\gamma, 0)$. Furthemore, using (2) we see $P_{T(S)}(T(\gamma, c - 1)) = (\gamma, 0)$ and, since $T(\gamma, c - 1)$ is the bottom of $T(S)$, the sphere $T(S)$
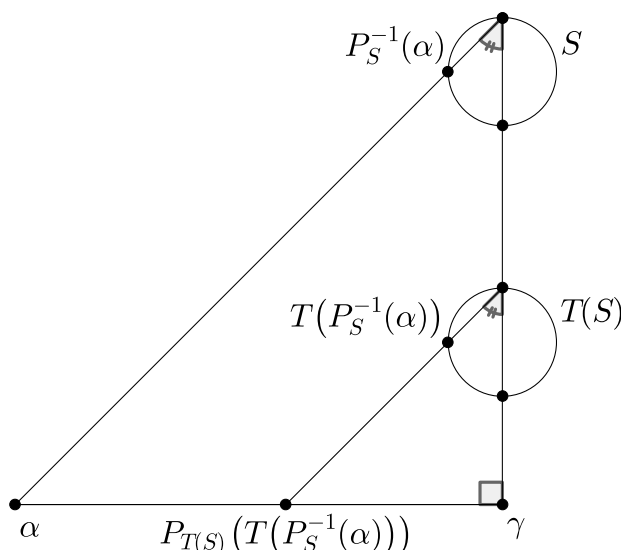
Figure 5: A cross-sectional view of S and $T(S)$ for a vertical translation T.

must also be centered above the point γ in the complex plane. Thus the two spheres S and $T(S)$ are centered on the same vertical line through $(\gamma, 0)$, and T is at most a vertical translation and rotation about that line.

To prove T does not incorporate a translation, consider a point $\alpha \neq \gamma \in \mathbb{C}$. Any vertical shift would result in $P_{T(S)}\left(T\left(P_S^{-1}(\alpha)\right)\right) \neq \alpha$ because the distance from $P_{T(S)}\left(T\left(P_S^{-1}(\alpha)\right)\right)$ to γ would not equal $|\gamma - \alpha|$, as can be easily verified using elementary triangle geometry; see Figure 5 for a cross-sectional representation of an example where T incorporates a downward vertical translation. The reader can verify that the two triangles in the figure are similar but not congruent, and thus have bases of different length.

The only remaining possibility for the rigid motion T is a rotation about the vertical axis of the sphere S, but the reader can quickly verify that any non-trivial rotation will contradict the assumption that $P_{T(S)} \circ T \circ P_S^{-1} = id$. Having ruled out all other possibilities we conclude that $T = id$, as desired.

$\square$

Having proven Lemma 5, the full proof of uniqueness in Theorem 4 follows quickly.

Lemma 6. *Given admissible sphere S and admissible rigid motions T, $\hat{T}$, if*

$$P_{T(S)} \circ T \circ P_S^{-1} = P_{\hat{T}(S)} \circ \hat{T} \circ P_S^{-1}$$

then $T = \hat{T}$.

We can compose the above maps on the right with $P_S \circ T^{-1} \circ P_{T(S)}$ and simplify to get

$$id = P_{\hat{T}(S)} \circ \hat{T} \circ T^{-1} \circ P_{T(S)}$$

As T and $\hat{T}$ are admissible, $T(S)$ and $\hat{T}(S)$ are admissible spheres. As $\hat{T} \circ T^{-1}$ maps $T(S)$ to the admissible sphere $\hat{T}(S)$, it is an admissible rigid motion. By Lemma 5, $\hat{T} \circ T^{-1} = I$, which implies that $T = \hat{T}$, proving uniqueness.

3.2 Existence

We now show that, given some admissible sphere, any Möbius Transformation can be modeled by inverse stereographic projection onto that sphere, followed by an admissible rigid motion, and stereographic projection back to the extended plane. By Lemma 6 this representation is necessarily unique, thus establishing Theorem 4.

Lemma 7. *Given Möbius Transformation f and admissible sphere S, there exists an admissible rigid motion T such that $f = P_{T(S)} \circ T \circ P_S^{-1}$.*

Recall from Section 2.1 that a Möbius Transformation is uniquely determined by its effects on three points. To prove the existence of T we construct a rigid motion so that $P_{T(S)} \circ T \circ P_S^{-1}$ sends the appropriate points to 0, 1, and ∞.

Proof. Consider the preimages under f of 0, 1, and ∞ and copy them onto the sphere S via inverse stereographic projection:

$$R = P_S^{-1}\left(f^{-1}(0)\right)$$
$$G = P_S^{-1}\left(f^{-1}(1)\right)$$
$$B = P_S^{-1}\left(f^{-1}(\infty)\right)$$

As the composition of rigid motions is a rigid motion, we will construct T piece by piece, following the projections of R, G, and B to ensure they are projected to 0, 1, and ∞. In particular, in what follows, R, G and B will refer to the transformation of the original points under any rigid motions which have been performed, and are depicted as red, green and blue points, respectively, in the figures below.

We begin with any rotation of the sphere which brings B to the top, followed by a horizontal translation by $-P_S(R)$. This ensures that B and R are sent to ∞ and the origin, respectively. This leads to two cases:

Case 1 . B and R may be antipodal, in which case they are necessarily on the x_3 axis of $\mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3$ at this step; see Figure 6a. We may translate the sphere up or down until the projection of G lies on the unit circle in $\mathbb{C}$. Finally, an appropriate rotation of the sphere about its vertical axis will move the projection of G to 1 on the positive real axis.

Case 2 B and R are not antipodal, in which case B, R and the origin are necessarily collinear; see Figure 6b. If the ray from the origin through B is rotated about the vertical axis it forms the yellow cone in the diagram. By sliding B and G along the ray, we can translate the sphere diagonally until G is projected onto the unit circle in

$\mathbb{C}$. Finally, we rotate the sphere around the vertical axis of $\mathbb{C} \times \mathbb{R}$, always leaving B and G on a ray from the origin (i.e. on the cone) until the projection of G reaches 1 on the positive real axis.

In both cases, the rigid transformations are described intuitively, but the rotations and translations can of course be explicitly determined using trigonometry and similar triangles.

By composing all of the above rigid transformations we have constructed an isometry T such that

$$P_{T(S)} \circ T \circ P_S^{-1} \left(f^{-1}(0)\right) = 0$$
$$P_{T(S)} \circ T \circ P_S^{-1} \left(f^{-1}(1)\right) = 1$$
$$P_{T(S)} \circ T \circ P_S^{-1} \left(f^{-1}(\infty)\right) = \infty$$

By Theorem 3, $P_{T(S)} \circ T \circ P_S^{-1}$ is a Möbius transformation; furthermore, we have shown it agrees with the values of f at three points. Since Möbius transformations are uniquely determined by their effect on any three points, we conclude

$$f = P_{T(S)} \circ T \circ P_S^{-1}$$

as desired, completing the proof.

$\square$

4   Future Directions

Although this paper answers the main open question in [2], other questions remain about this method of constructing Möbius transformations. For example, in the existence proof, we characterized the rigid motion required to construct a specific Möbius transformation

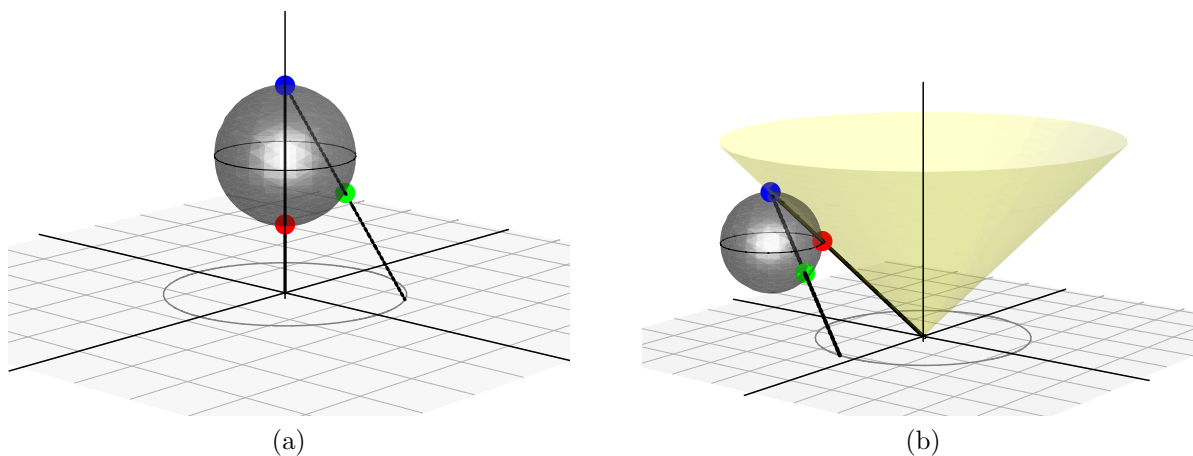(a)   (b)

Figure 6: Two cases in the proof of Lemma 7

for a given admissible sphere, but a different admissible sphere would require a different rigid motion. For a specified Möbius transformation f, we know of no relationship between the (infinitely many) rigid motions used to construct f for all possible admissible spheres. Further work could provide such a characterization.

References

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[3] Stephen Fisher. *Complex Variables.* Dover Publications, 1999.