Riemann-Stieltjes Integrals - Integration and Differentiation

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- Till now we dealt with some of the properties of the Riemann-Stieltjes integrals.
- In this lesson, we make an attempt to exploit our knowledge of derivatives to compute the integrals.
- The results we prove can be seen as a useful tool that provides us some of the nice properties that we seek.

- We start with the Mean-Value theorem for integrals, which reads

Theorem

Suppose that f is continuous on $I = [a, b]$, then there exists a number $s \in I$ such that

$$\int_a^b f(x)dx = f(s)(b-a).$$

Mean-Value Theorem.

Since f is continuous on $[a, b]$, we have

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

where $m = min_I f(x) = f(x')$ and $m = max_I f(x) = f(x'')$, $x', x'' \in I$. Now,

$$\gamma = \frac{\int_a^b f(x)dx}{b-a}$$

is a real number such that $f(x') = m \leq \gamma \leq M = f(x'')$. Then, Intermediate mean value theorem implies that there exist an $s \in I$ such that

$$f(s) = \gamma = \frac{\int_a^b f(x)dx}{b-a} \text{ and the result follows.}$$

- The next two results, namely, the fundamental theorems we prove are two of the most celeberated results.
- They draw a clear and important connection between integral and differential calculus.
- The first one of the two allows us to define new functions in terms of integrals.
- This function is some times referred to as accumultation of f.

Theorem

Suppose that $f \in \mathcal{R}$ *on* $[a, b]$. *Then the function* F *given by*

$$F(x) = \int_a^x f(t)dt$$

is uniformly continuous on $[a, b]$. *If* f *is continuous on* $[a, b]$, *then* F *is differentiable in* (a, b) *and for each* $x \in (a, b)$, $F'(x) = f(x)$.

Proof- Part a.

Let $u, v \in [a, b]$. Without loss of generality we may assume that $u < v$. Then consider

$$\begin{aligned}
|F(v) - F(u)| &= |\int_a^v f(t)dt - \int_a^u f(t)dt|, \\
&= |\int_a^u f(t)dt + \int_u^v f(t)dt - \int_a^u f(t)dt|, \\
&= |\int_u^v f(t)dt|, \\
&\leq M|v - u|,
\end{aligned}$$

$$\therefore \forall u, v \in [a, b], \;\; |v - u| < \delta = \frac{\epsilon}{M} \;\Rightarrow\; |F(v) - F(u)| < \epsilon.$$

Hence, F is uniformly continuous on $[a, b]$. □

Proof- Part b.

In order to prove the second part, suppose that f is continuous and $x \in (a, b)$. Then there exist a δ_1 such that $\{x + h : |h| < \delta_1\} \subset (a, b)$. Now f being continuous it is integrable on every sub-interval of $[a, b]$. From this it follows that each of

$$\int_a^{x+h} f(t)dt, \quad \int_a^{x} f(t)dt \text{ and } \int_x^{x+h} f(t)dt$$

exists for $|h| < \delta_1$, and in that case

$$F(x + h) - F(x) = \int_x^{x+h} f(t)dt, \quad \text{for any } h \text{ with } |h| < \delta_1 \ .$$

□

Proof- Part b Continues.

By the Mean Value theorem for integral, there exists ξ_h with $|x-\xi_h|<\delta_1$ so that the following holds

$$\int_x^{x+h} f(t)dt = hf(\xi_h).$$

Combining the last two equalities, we have

$$\frac{F(x+h)-F(x)}{h} = f(\xi_h), \quad \text{where } |x-\xi_h|<\delta_1.$$

□

Proof- Part b Continues.

Further assume that $\epsilon > 0$ is given. Since f is continuous at x, there exists a $\delta_2 > 0$ such that

$$|f(k) - f(x)| < \epsilon \quad \text{whenever } |k - x| < \delta_2.$$

Choose $\delta = min\{\delta_1, \delta_2\}$. Then, for $h < \delta$, we have

$$|\frac{F(x+h) - F(x)}{h} - f(x)| = |f(\xi_h) - f(x)| < \epsilon.$$

Since $\epsilon > 0$ was chosen arbitrarily. It follows that

$$F'(x) = f(x).$$

Now $x \in (a, b)$ was arbitrary, it follows that F is differentiable on the open interval (a, b). □

- Note that the statement of the first fundamental theorem of calculus differs from the one that we used to study in elementary calculus.
- What is the difference?

- We now try to offer a slightly different proof for the second fundamental theorem of calculus. The theorem reads

Theorem

Suppose that $f \in \mathcal{R}$ on $[a, b]$, and there is a function F that is differentiable on $[a, b]$ with $F' = f$, then

$$\int_a^b f(t)dt = F(b) - F(a)$$

.

Proof.

Let $\epsilon > 0$ is given. Since $f \in \mathcal{R}$ on $[a, b]$ there exists a partition $P \in \mathcal{P}[a, b]$ such that

$$U(P, f) - L(P, f) < \epsilon, \quad \text{and}$$

$$\left|\sum_{j=1}^{n} f(t_j)\Delta x_j - \int_a^b f dx\right| < \epsilon. \tag{1}$$

By Mean Value theorem, for each $j \in \{1, 2, \dots n\}$ there is a point $t_j \in [x_{j-1}, x_j]$ such that

$$\frac{F(x_j) - F(x_{j-1})}{x_j - x_{j-1}} = F'(t_j),$$

$$\text{i.e.,} \quad f(t_j)\Delta x_j = F(x_j) - F(x_{j-1}).$$

Proof Continues.

$$\text{i.e., } \sum_{j=1}^{n} f(t_j)\Delta x_j = \sum_{j=1}^{n}[F(x_j) - F(x_{j-1})], \\ = F(b) - F(a).$$

Therefore from equation (1), we have

$$|F(b) - F(a) - \int_a^b f dx| < \epsilon.$$

Since $\epsilon > 0$ was arbitrary it follows that

$$F(b) - F(a) = \int_a^b f dx.$$

- We now establish an elementary result which is an immediate consequence of the results we just proved.

Theorem

If F and G are differentiable functions on $[a, b]$ and $F' = f \in \mathcal{R}$ and $G' = g \in \mathcal{R}$ then

$$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b G(x)f(x)dx.$$

Proof.

Put H(x)=F(x)G(x). Then differentiation of H leads to

$$H'(x) = F'(x)G(x) + F(x)G'(x) = f(x)G(x) + F(x)g(x).$$

Now, F and G being differentiable are continuous. Therefore F and G are Riemann integrable. Algebraic property of Riemann integrable function gives that,

$$F.G, \quad Fg + Gf \in \mathcal{R}, \quad \text{and hence} \quad H'(x) \in \mathcal{R}.$$

□

Proof Continues.

An application of Fundamental Theorem of integral calculus yields

$$\int_a^b [F(x)g(x) + G(x)f(x)]dx = H(b) - H(a),$$

i.e., $$\int_a^b F(x)g(x)dx + \int_a^b G(x)f(x)dx = F(b)G(b) - F(a)G(a),$$

i.e., $$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b G(x)f(x)dx.$$

□

Definition

Let $f_1, f_2, \ldots, f_n$ be real valued bounded functions defined on $[a, b]$. Given a vector valued function $\mathbf{f} = (f_1, f_2, \ldots, f_n)$ from $[a, b]$ into $\Re^n$ and a monotonically increasing function α that is defined on $[a, b]$. Then $\mathbf{f}$ is Riemann Stieltjes integrable with respect to α on $[a, b]$, written $\mathbf{f} \in \mathcal{R}(\alpha)$, if and only if $f_j \in \mathcal{R}(\alpha)$, $\forall j \in (1, 2, \ldots, n)$. In this case

$$\int_a^b \mathbf{f}(x) d\alpha(x) = \Big(\int_a^b f_1(x) d\alpha(x), \int_a^b f_2(x) d\alpha(x), \ldots \\ \ldots, \int_a^b f_n(x) d\alpha(x) \Big).$$

Theorem

Suppose that the vector valued functions $\mathbf{f}$ *and* $\mathbf{g}$ *are Riemann Stieltjes integrable with respect to* α *on the interval* $[a, b]$ *and* k *is any real constant, then*

1. $\mathbf{f} + \mathbf{g} \in \mathcal{R}(\alpha)$ *on* $[a, b]$ *and*

$$\int_a^b (\mathbf{f} + \mathbf{g}) d\alpha = \int_a^b \mathbf{f} d\alpha + \int_a^b \mathbf{g} d\alpha,$$

2. $k\mathbf{f} \in \mathcal{R}(\alpha)$ *on* $[a, b]$ *and*

$$\int_a^b k\mathbf{f} d\alpha = k \int_a^b \mathbf{f} d\alpha.$$

Theorem

Suppose that $\mathbf{f}, \mathbf{g} \in \mathcal{R}(\alpha)$ *on* $[a, b]$ *and*

1. *if the function* $\mathbf{f} \in \mathcal{R}(\alpha)$ *also on* $[b, c]$, *then* $\mathbf{f}$ *is Riemann Stieltjes integrable with respect to* α *on* $[a, b] \cup [b, c]$ *and*

$$\int_a^c \mathbf{f}(x)\alpha(x) = \int_a^b \mathbf{f}(x)\alpha(x) + \int_b^c \mathbf{f}(x)\alpha(x).$$

2. *if* k *be any positive real constant, then* $\mathbf{f} \in \mathcal{R}(k\alpha)$ *and*

$$\int_a^b \mathbf{f}(x)d\alpha(kx) = k\int_a^b \mathbf{f}(x)d\alpha(x).$$

Theorem

Suppose that α is a monotonically increasing function such that $\alpha' \in \mathcal{R}$ on $[a, b]$ and $\mathbf{f}$ is a vector valued bounded function that is defined on $[a, b]$ into $\Re^n$. Then $\mathbf{f} \in \mathcal{R}(\alpha)$ if and only if $\mathbf{f}\alpha' \in \mathcal{R}$. Furthermore,

$$\int_a^b \mathbf{f}(x)d\alpha(x) = \int_a^b \mathbf{f}(x)\alpha'(x)dx.$$

Theorem

Suppose that $\mathbf{f} = (f_1, f_2, \ldots, f_n) \in \mathcal{R}$ *on* $[a, b]$.

1. *Then the vector valued function* $\mathbf{F}$ *given by*

$$\mathbf{F}(x) = \left(\int_a^x f_1(t)dt, \int_a^x f_2(t)dt, \ldots, \int_a^x f_n(t)dt, \right); \; x \in [a, b]$$

   is continuous on $[a, b]$. *Furthermore if* $\mathbf{f}$ *is continuous on* $[a, b]$, *then* $\mathbf{F}$ *is differentiable in* (a, b) *and, for each* $x \in (a, b)$,

$$\mathbf{F}'(x) = \mathbf{f}(x) = (f_1(x), f_2(x), \ldots, f_n(x)).$$

2. *If there exists a vector valued function* $\mathbf{G}$ *on* $[a, b]$ *that is differentiable there with* $\mathbf{G}' = \mathbf{f}$, *then* $\int_a^b \mathbf{f}(t)dt = \mathbf{G}(b) - \mathbf{G}(a)$.

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Thank You !