Riemann Stieltjes Integration - Properties

Dr. Aditya Kaushik

Directorate of Distance Education
Kurukshetra University, Kurukshetra Haryana 136119 India.

Till now, we are playing with the functions which are smooth enough or continuous in precise sense. What if we are given with the functions which are not continuous? The next theorem answers this question and relates us to a class of discontinuous function which are integrable in the sense of Riemann Stieltjes.

Theorem

Suppose that f is bounded on $[a, b]$, f has only finitely many points of discontinuity in $I = [a, b]$, and that the monotonically increasing function α is continuous at each point of discontinuity of f. Then $f \in \mathcal{R}(\alpha)$.

Proof.

Let $\epsilon > 0$ be given. Suppose that f is bounded on $[a, b]$ and continuous on $[a, b] - I$ where $I = [a_1, a_2, \ldots, a_p]$ is the nonempty finite set of points of discontinuity of f in $[a, b]$. Suppose further that α is monotonically increasing function on $[a, b]$ that is continuous at each element of I. Because I is finite and α is continuous at each $a_j \in I$, we can find p pairwise disjoint intervals $[u_j, v_j]$, $j = 1, 2, \ldots, p$, such that

$$I \subset \bigcup_{j=1}^{p} [u_j, v_j] \subset [a, b] \quad \text{and} \quad \sum_{j=1}^{p} (\alpha(v_j) - \alpha(u_j)) < \epsilon^*.$$

Further place these intervals in such a way that each $a_j \in I \bigcap (a, b)$ lies in the interior os some $[u_j, v_j]$. □

Proof Continues.

Remove the segment (u_j, v_j) from $[a, b]$. Then the remaining set

$$\begin{aligned} K &= [a,b] - \bigcup_{j=1}^{p} (u_j, v_j) \\ &= [a, u_1] \bigcup [v_1, u_2] \bigcup \ldots \bigcup [v_p, b] \end{aligned}$$

is compact. Moreover f is uniformly continuous on K. Therefore, corresponding to ϵ^* there is a $\delta > 0$ such that

$$|f(s) - s(t)| < \epsilon^* \text{ whenever } |s - t| < \delta, \quad \forall s, t \in K.$$

□

Proof Continues.

Now, form a partition $P = \{a = x_0, x_1, ...x_{n-1}, x_n = b\}$ of $[a, b]$ in such a way that the following properties holds;

1. $u_j \in P$, $\forall j \in \{1, 2, \ldots p, \}$
2. $v_j \in P$, $\forall j \in \{1, 2, \ldots p, \}$
3. $(u_j, v_j) \bigcap P = \phi$, $\forall j \in \{1, 2, \ldots p, \}$ and
4. $x_{i-1} \neq u_j \Rightarrow \Delta x_i < \delta$, $\forall i \in \{1, 2, \ldots n, \}$ and $\forall j \in \{1, 2, \ldots p.\}$

□

Proof Continues.

It is easy to follow under the conditions establised that $x_{i-1} = u_j$ implies $x_i = v_j$. Define $M = sup_x|f(x)|$, $M_i = sup_{x\in[x_{i-1},x_i]}|f(x)|$ and $m_i = inf_{x\in[x_{i-1},x_i]}|f(x)|$. Then for each i,

$$0 \leq M_i - m_i = |M_i - m_i| \leq |M_i| + |m_i| \leq M + M = 2M.$$

Further as long as $x_{i-1} \neq u_j$, we have $\Delta x_i = |x_i - x_{i-1}| < \delta$,

$$\Rightarrow |f(x_i) - f(x_{i-1})| < \epsilon^*, i.e. |M_i - m_i| < \epsilon^*.$$

□

Proof Continues.

Consider,

$$\begin{aligned} U(P,f,\alpha) - L(P,f,\alpha) &= \sum_{j=1}^{n}(M_j - m_j)\Delta\alpha_j \\ &\leq \epsilon^*[\alpha(b) - \alpha(a)] + 2M\epsilon^* \\ &< \epsilon \end{aligned}$$

where $\epsilon^* < \frac{\epsilon}{[\alpha(b)-\alpha(a)]+2M}$. Because $\epsilon > 0$ was chosen arbitrary $f \in \mathcal{R}(\alpha)$ follows from integrability criterion. □

It is important to note that in case f and α have common point of discontinuity then it is not necessary that f is Riemann-Stieltjes integrable.

The following theorem provides us the sufficient condition for the composition of a function with a Riemann-Stieltjes integrable function to be Riemann-Stieltjes integrable.

Theorem

Suppose $f \in \mathcal{R}(\alpha)$ *on* $[a, b]$ *such that* $m \leq f \leq M$ *and* ϕ *a continuous function defined on* $[m, M]$. *If* $h = \{\phi(f(x)) : x \in [a, b]\}$ *then* $h \in \mathcal{R}(\alpha)$ *on* $[a, b]$.

The above theorem together in combination with fundamental properties of Riemann-Stieltjes integrals, allows us to generate a set of Riemann-Stieltjes integrable functions. This is the context of next theorem.

Theorem

Let f be a bounded real valued function such that $f \in \mathcal{R}(\alpha)$ on $[a, b]$, then $f^2,\ |f| \in \mathcal{R}(\alpha)$ on $[a, b]$.

Proof.

Take $\phi_1(t) = t^2$ and $\phi_2(t) = |t|$. Because both $\phi_1(t)$ and $\phi_2(t)$ is continuous and $f \in \mathcal{R}(\alpha)$ on $[a, b]$. Theorem 2 asserts that $\phi_1(f) = f^2 \in \mathcal{R}(\alpha)$ and $\phi_2(f) = |f| \in \mathcal{R}(\alpha)$ on $[a, b]$. □

This part offers a list of algebraic properties of Riemann-Stieltjes integrals.

Theorem

Suppose that $f, g \in \mathcal{R}(\alpha)$ *on* $[a, b]$ *and* k *be any constant, then*

1. $f + g \in \mathcal{R}(\alpha)$ *on* $[a, b]$ *and*

$$\int_a^b (f+g)d\alpha = \int_a^b f d\alpha + \int_a^b g d\alpha,$$

2. $kf \in \mathcal{R}(\alpha)$ *on* $[a, b]$ *and*

$$\int_a^b kf d\alpha = k \int_a^b f d\alpha.$$

Theorem

Let $f \in \mathcal{R}(\alpha)$ and $g \in \mathcal{R}(\alpha)$ on $[a, b]$, then $fg \in \mathcal{R}(\alpha)$ on $[a, b]$.

Proof.

Suppose that $f, g \in \mathcal{R}(\alpha)$ on $[a, b]$. From the Algebraic Properties of the Riemann-Stieltjes Integral, it follows that $(f + g), (f - g) \in \mathcal{R}(\alpha)$ on $[a, b]$. Theorem 3 then gives that

$$(f + g)^2, (f - g)^2 \in \mathcal{R}(\alpha) \text{ on } [a, b].$$

Therefore, constant times the difference

$$fg = \frac{1}{4}[(f + g)^2 - (f - g)^2] \in \mathcal{R}(\alpha) \text{ on } [a, b].$$

□

Lemma

Suppose that f *is bounded on* $[a, b]$ *and continuous at* $s \in (a, b)$. *If* $\alpha(x) = \chi(x - s)$, *then*

$$\int_a^b f d\alpha = f(s).$$

- In addition, if f is continuous on $[a, b]$ then it is possible to extend above Lemma 6 to a sequence of points in that interval. This is the context of our next result we prove.

Theorem

Suppose $< c_n >_{n=1}^{\infty}$ *be a sequence of non-negative real numbers such that* $\sum_{n=1}^{\infty} c_n$ *is convergent, a sequence* $< s_n >_{n=1}^{\infty}$ *of distinct point in* (a, b) *and* f *is continuous on* $[a, b]$. *If* $\alpha(x) = \sum_{n=1}^{\infty} c_n \chi(x - s_n)$, *then*

$$\int_a^b f d\alpha = \sum_{n=1}^{\infty} c_n f(s_n).$$

Proof.

From the definition of α it is clear that $\alpha(a) = 0$ and $\alpha(b) = \sum_{n=1}^{\infty} c_n$, because $a \leq s_n$ and $b \geq s_n$ for every $n \in N$. Let $u, v \in (a, b)$ be such that $u < v$ and define

$$A = \{n \in N : a < s_n \leq u\} \text{ and } B = \{n \in N : a < s_n \leq v.\}$$

□

Proof Continues.

Then $A \subseteq B$ since $u < v$ and therefore

$$\alpha(u) = \sum_{n \in A} c_n \leq \sum_{n \in B} c_n = \alpha(v)$$

from this it follows that α is monotonicaly increasing. Let $\epsilon > 0$ be given and $M = sup_{x \in [a,b]} |f(x)|$. Because $\sum_{n=1}^{\infty} c_n$ is convergent, there exists a positive integer k such that

$$\sum_{n=k+1}^{\infty} c_n < \frac{\epsilon}{M}. \tag{1}$$

□

Proof Continues.

Decompose α into two parts so that $\alpha = \alpha_1 + \alpha_2$ where $\alpha_1 = \sum_{n=1}^{k} c_n \chi(x - s_n)$ and $\alpha_2 = \sum_{n=k+1}^{\infty} c_n \chi(x - s_n)$. It follows from Lemma 6 that

$$\begin{aligned} \int_a^b f d\alpha_1 &= \int_a^b f d[\sum_{n=1}^{k} c_n \chi(x - s_n)], \\ &= c_1 \int_a^b f d[\chi(x - s_1)] + \ldots + c_k \int_a^b f d[\chi(x - s_k)], \\ &= c_1 f(s_1) + \ldots + c_k f(s_k), \\ &= \sum_{n=1}^{k} c_n f(s_n). \end{aligned} \tag{2}$$

□

Proof Continues.

Now definition of α together with (1) yields

$$\begin{aligned}\alpha_2(b)-\alpha_2(a) &= \sum_{n=k+1}^{\infty} c_n\chi(b-s_n) - \sum_{n=k+1}^{\infty} c_n\chi(a-s_n)\\ &= \sum_{n=k+1}^{\infty} c_n < \frac{\epsilon}{M}.\end{aligned}$$

Consequently, it gives

$$\begin{aligned}\left|\int_a^b f d\alpha_2\right| &\leq M[\alpha_2(b)-\alpha_2(a)],\\ &\leq \epsilon. \qquad (3)\end{aligned}$$

□

Proof Continues.

Now consider

$$
\begin{aligned}
\left|\int_a^b f d\alpha - \sum_{n=1}^{k} c_n f(s_n)\right| &= \left|\int_a^b f d(\alpha_1+\alpha_2) - \sum_{n=1}^{k} c_n f(s_n)\right|, \\
&= \left|\int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 - \sum_{n=1}^{k} c_n f(s_n)\right|, \\
&= \left|\int_a^b f d\alpha_2\right|, \\
&< \epsilon.
\end{aligned}
$$

□

Proof Continues.

Taking limit as $k \to \infty$ both the sides and using the fact that modulus is continuous. This leads to

$$|\int_a^b f d\alpha - lim_{k\to\infty} \sum_{n=1}^{k} c_n f(s_n)| < \epsilon,$$

$$\text{i.e.,} \quad |\int_a^b f d\alpha - \sum_{n=1}^{\infty} c_n f(s_n)| < \epsilon.$$

Since $\epsilon > 0$ was arbitrary $\int_a^b f d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$. □

1. Walter Rudin : Principles of Mathematical Analysis, McGraw Hill Pulishers.
2. T. Apostol, Mathematical Analysis, Narosa Publication.
3. A. Kaushik, Lecture Notes, Directorate of Distance Education, Kurukshetra University Kurukshetra.

Thank You !