MA5360 – Assignment 2
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1. Formulate a version of chain rule for composition of $\mathbb{C}$-differentiable functions and give a short proof. What can you tell about composition of complex-analytic functions?

2. Let $D \subset \mathbb{C}$ be a domain and suppose $f \in H(D)$. Show that

   a) if $\overline{f} \in H(D)$ then f is constant;
   b) if $|f|$ is constant then so is f;
   c) if $f = u + iv$ then $\triangle u = \triangle v \equiv 0$. Here $\triangle$ denotes the Laplacian.
   d) $\overline{f(\overline{z})}$ is holomorphic on the domain $D^* := \{\overline{z} : z \in D\}$.

3. Show that the function $f(z) = \sqrt{|xy|}$ satisfies the Cauchy–Riemann equations at 0 but is *not* $\mathbb{C}$-differentiable at 0.

4. Let $f(z) = \sum c_n z^n$ be a power series that converges in $D(a, R), R > 0$ and $f'(0) = c_1 \neq 0$. Prove the equality

$$|f(z) - f(w)| = |z - w| \left| \sum c_n (z^{n-1} + z^{n-2}w + \cdots + zw^{n-2} + w^{n-1}) \right|$$

   for all $z, w \in D(0, R)$ and consequently that f is injective on the disc $D(0, r)$ if $0 < r < R$ and the following inequality holds:

$$\sum_{n=2}^{\infty} n|c_n|r^{n-1} < |c_1|.$$

5. Let $f(z) = \sum c_n z^n$ be a power series that converges in $D(a, R), R > 0$ and suppose $0 < r < R$. Show:

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it}|^2 dt = \sum |c_n|^2 r^{2n}.$$

   Use this equality to deduce that a bounded complex-analytic function on $\mathbb{C}$ must be constant.

6. Show that the function $\cos z$ maps the strip $B = \{z : 0 < \mathrm{Re} z < \pi\}$ onto the domain $U = \mathbb{C} \setminus \{x \in \mathbb{R} : |x| \geqslant 1\}$ conformally and injectively. Find an expression for its inverse in terms of the logarithm.

7. Look up the definition of real-analytic functions in $\mathbb{R}^n$. Let $f : U \to \mathbb{C}$ be a complex-analytic function. Are $\mathsf{Re} f$ and $\mathsf{Im} f$ real-analytic?

8. Let $f(z) = \sum c_n z^n$ be a convergent power series on $D := D(0, R)$. Show that for each $0 < r < R$ and $m > 1$, $|f^{(m)}(0)| \leqslant m! M(r)/r^m$ where $M(r) = \sup\{|f(z)| : |z| = r\}$.