MA5360 – Assignment 1
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1. Prove the reverse triangle inequality: $|w - z| \geqslant ||w| - |z||$ for all $z, w \in \mathbb{C}$ with equality iff either z or w is real positive multiple of the other.
2. Determine all positive integers n for which i is an n-th root of unity.
3. Let $n \geqslant 1$ and let $\omega_0, \ldots, \omega_{n-1}$ be the n-th roots of unity. Show that
   a) $(z-\omega_0)(z-\omega_1)\cdots(z-\omega_{n-1}) = z^n - 1$;
   b) $\omega_0 + \cdots + \omega_{n-1} = 0$ if $n > 1$;
   c) $\omega_0 \times \cdots \times \omega_{n-1} = (-1)^{n-1}$;
   d) $\sum_{j=0}^{n-1} \omega_j^k = \begin{cases} 0, & 1 \leqslant k \leqslant n-1 \\ n, & k = n. \end{cases}$
4. Given an example of a Jordan loop of infinite length.
5. Show that any path $\gamma : [a, b] \to \mathbb{C}$ is rectifiable and that
$$L(\gamma) = \int_a^b |\gamma'(t)| dt.$$
   `Hint:` Use the mean value theorem.
6. Show that if there is a branch of $\sqrt{z}$ on an open set U with $0 \notin U$, then there is also a branch of $\arg z$.
7. Let γ parametrize a square traversed counter-clockwise and z be a point in the interior of the square. Show that $\text{Ind}(\gamma, z) = 1$. What if γ was clockwise?
8. Let $\gamma : [a, b] \to \mathbb{C}$ be a curve. Show that $\text{Ind}(\gamma, z)$ is continuous on $\mathbb{C} \setminus \gamma^*$.
9. Let $\gamma_1 : [a, b] \to \mathbb{C}$ and $\gamma_2 : [c, d] \to \mathbb{C}$ be two curves such that $\gamma_1(b) = \gamma_2(c)$. Define a natural way to "add" these curves to produce a third curve γ_3. Now if $z \in \mathbb{C} \setminus \gamma_3^*$, write down an expression for $\text{Ind}(\gamma_3, z)$ in terms of $\text{Ind}(\gamma_1, z)$ and $\text{Ind}(\gamma_2, z)$.
   `Hint:` You can assume, without loss of generality, that $c = b$.
10. Let $\gamma : [a, b] \to \mathbb{C}$ be a closed curve that misses 0 and let $z_1 = \gamma(t_1)$ and $z_2 = \gamma(t_2)$ be two points on the curve with $a < t_1 < t_2 < b$. Let $\gamma_1 := \gamma|_{[t_1, t_2]}$ be the sub-arc on the curve from z_1 to z_2 and let $\gamma_2 := \gamma|_{[t_2, b]} + \gamma|_{[a, t_1]}$ be the sub-arc from z_2 to z_1. Suppose that z_1 lies in the lower half plane and z_2 in the upper half plane. If γ_1 does not meet the negative real axis and γ_2 does not meet the positive real axis, prove that $\text{Ind}(\gamma, 0) = 1$.