Harmonic Functions

In this short chapter, we study harmonic functions. The study of harmonic functions originally arose from physics but our interest in them stems from the fact that the real and imaginary parts of holomorphic functions are harmonic. We will study the basic properties of harmonic functions in this chapter. For a more extensive treatment, check any good textbook on potential theory (I recommend Ransford's textbook.)

1 Definition and basic properties

Definition 1. Let $U \subset \mathbb{C}$ be open. A function $u : U \to \mathbb{R}, u \in \mathcal{C}^2(U)$ is said to be *harmonic* if the laplacian of u vanishes, i.e., if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

It is immediate from the CR-equations that the real and imaginary parts of holomorphic functions are automatically harmonic. A more complicated example follows:

Example 2. Let $f : U \to \mathbb{R}$ be holomorphic and suppose that $f(z) \neq 0 \forall z \in U$. Then $\log|f|$ is harmonic. To see this, first observe that being harmonic is a local property and therefore it suffices to consider a point $z \in U$ and prove that $\log|f|$ is holomorphic in some open neighbourhood of z. As $f(z) \neq 0$, we can find a small neighbourhood $V \subset U$ such that $f(V)$ is contained in a small disk that misses 0. On this disk, we can find a continuous, and therefore holomorphic branch of the logarithm, say h. Now, $h \circ f$ is holomorphic on V and as h is branch of the logarithm, it follows that $\operatorname{Re} h \circ f = \log|f|$. This proves that $\log|f|$ is harmonic on V and therefore, $\log|f|$ is harmonic on U.

Our strategy in the above example was to locally exhibit $\log|f|$ as the real part of a holomorphic function. The following theorem says that this can be done for all harmonic functions

Theorem 3. *Let $u : U \to \mathbb{R}$ be a harmonic function. Then for each $z \in U$, we can find a neighbourhood V of z and a holomorphic function f on V such that $u = \operatorname{Re} f$.*

Proof. let $g := u_x - iu_y$. Then the fact that u is harmonic and the equality of mixed partial derivatives imply that $g \in H(U)$. Let $z \in U$ and let V be a small disk centred at z and fully contained in U. On this disk V, g has an anti-derivative, say f. Let $\operatorname{Re} f = h$. Then $h_x = u_x$ and $h_y = u_y$ because $f' = g$. From this it follows that $u - h$ is a constant c. It follows that $g - c$ is the required holomorphic function. $\qquad\square$

Corollary 4. *Harmonic functions are $\mathcal{C}^\infty$ on their domain of definition.*

Definition 5. Let $u : U \to \mathbb{R}$ be harmonic. We say that a harmonic function $v : U \to \mathbb{R}$ is a *harmonic conjugate* for u if $u + iv$ is holomorphic.

Remark 6. It is easy to see that harmonic conjugates are unique up to a constant. A harmonic function need not necessarily have a harmonic conjugate on U. Take, for instance, $\log z$ which is harmonic on $\mathbb{C} \setminus 0$.

Remark 7. It is interesting to determine on which domains U, *every* harmonic function has a harmonic conjugate. These are precisely the *simply-connected* domains. These are also precisely the domains on which every nowhere vanishing holomorphic function has a continuous branch of the logarithm. An *ad-hoc* definition of simply-connected is as follows: A domain U is said to be *simply-connected* if $\mathbb{C}^* \setminus U$ is connected.

2 The maximum principle

We will prove several facts about harmonic functions that are consequences of their relationship with holomorphic functions.

Theorem 8 (The identity theorems). *Let U be a domain and suppose that u is a harmonic function on U that vanishes on an open subset V. Then $u \equiv 0$.*

Proof. The function $g := u_x - iu_y$ is holomorphic and identically 0 on V. By the principle of analytic continuation for holomorphic functions, it follows that $g \equiv 0$. This means that $u_x, u_y \equiv 0$ from which it follows that $u \equiv 0$. $\qquad\square$

Remark 9. The identity theorem or principle of analytic continuation for holomorphic functions says that a holomorphic function that vanishes on an indiscrete set is identically 0. This stronger statement is not true for harmonic functions: consider, for example, the functions $u(x, y) = x = \text{Re} z$.

Theorem 10 (Mean-value property). *Let $u : U \to \mathbb{R}$ be harmonic and suppose that $\overline{D}(a, R) \subset U$. Then*

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + Re^{it})dt = \frac{1}{\pi R^2} \iint_{D(a,R)} u(x, y)dxdy.$$

Proof. On the disk $D(a, R)$, we can find a holomorphic function f such that $\text{Re} f = u$. The result now follows immediately from the fact that holomorphic functions satisfy the mean value property. $\qquad\square$

Theorem 11 (The maximum principle). *Let U be a domain and suppose that u is harmonic on U. If u has a local-maximum on U then u is identically constant.*

Proof. Let $a \in U$ be a point of local-maximum and let $u(a) = M$. From the definition of local maxima, we can find a disk $\overline{D}(a, R) \subset U$ such that $u(z) \leqslant M$ on $D(a, R)$. We claim that $u \equiv M$ on $D(a, R)$. If not, we can find $b \in M$ and a disk $D(b, r) \subset D(a, R)$ such that $u < M$ on $D(b, r)$. From the mean value property, we see that

$$M = u(a) = \frac{1}{\pi R^2} \iint_{D(a,R)} u(x, y)dxdy$$

$$= \frac{1}{\pi R^2} \iint_{D(b,r)} u(x, y)dxdy + \iint_{D(a,R)\setminus D(b,r)} u(x, y)dxdy < \frac{1}{\pi R^2} M \times \pi R^2 = M,$$

which is a contradiction. This proves that $u \equiv M$ on $D(a, R)$ and thus on U by the identity theorem for harmonic functions. $\square$

Theorem 12. *Let* $u : \mathbb{C} \to \mathbb{R}$ *be harmonic and bounded. Then* u *is constant.*

Proof. Let $a, b \in \mathbb{C}$. The goal is to prove that $u(a0 = u(b)$. Choose $R > 0$ very large. From the mean value property, it follows that

$$u(a) = \frac{1}{\pi R^2} \iint u(x, y) \, dx \, dy,$$

and a similar expression for $u(b)$. This means that

$$u(a) - u(b) = \frac{1}{\pi R^2} \left[\iint_{D(a,R) \setminus D(b,R)} u(x, y) \, dx \, dy - \iint_{D(b,R) \setminus D(a,R)} u(x, y) \, dx \, dy \right]$$

If $R \to \infty$, it is easy to see that

$$\frac{\text{area}(D(b, R) \setminus D(a, R)) + \text{area}(D(a, R) \setminus D(b, R))}{\text{area}(D(a, R))} \to 0$$

This means that if M is the upper bound for u, then

$$u(a) - u(b) = 2M \times \text{quantity that goes to 0.}$$

This is proves $u(a) = u(b)$. $\square$