Complex Integration

As we have mentioned in the previous chapter, one of the central results of this course is the fact that holomorphic functions are in fact complex-analytic. To prove this we need to develop the machinery of complex integration. The main result is—just as in the case of real analysis—the link between integration and differentiation. We will prove a version of the fundamental theorem of calculus for complex line integrals. We will also prove a version of Green's theorem which will be the main tool used in proving the famous result of Cauchy on the vanishing of line integrals of holomorphic functions on closed paths.

Our approach will be through differential forms and vector fields. This will allow us to understand holomorphicity from the perspective of real analysis.

1 Complex line integrals

Let $f : [a, b] \to \mathbb{C}$ be continuous and let $f = u(t) + iv(t)$. We define

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt.$$

Let $\alpha, \beta \in \mathbb{C}$ and let $g : [a, b] \to \mathbb{C}$, then

$$\int_a^b (\alpha f + \beta g)dt = \alpha\int_a^b f(t)dt + \beta\int_a^b g(t)dt.$$

We have the following useful inequality:

$$\left|\int_a^b f(t)dt\right| \leqslant \int_a^b |f(t)|dt.$$

To see this, let $\int_a^b f(t)dt = re^{i\theta}$. Then

$$\begin{aligned}\left|\int_a^b f(t)dt\right| = r = e^{-i\theta}\int_a^b f(t)dt \\ &= \operatorname{Re}\left[e^{-i\theta}\int_a^b f(t)dt\right] = \int_a^b \operatorname{Re}\left[e^{-i\theta}f(t)\right] dt \\ &\leqslant \int_a^b |f(t)|dt.\end{aligned}$$

Definition 1. Let $\gamma : [a, b] \to \mathbb{C}$ be a $\mathcal{C}^1$ curve and let $f : \gamma^* \to \mathbb{C}$ be continuous. We define

$$\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Note that the above integral is the limit of the complex Riemann sums

$$\sum_i f(z_i)(z_{i+1} - z_i),$$

where the points $z_i := \gamma(t_i)$ are the vertices of a polygonal approximation of γ. Replacing $z_{i+1} - z_i$ by $|z_{i+1} - z_i|$ above, we recover the definition of integration with respect to arc-length

$$\int_\gamma f(z)|dz| := \int_\gamma f(z)ds.$$

It is clear that $\int_\gamma dz = \gamma(b) - \gamma(a)$.

Now, let $\Gamma = \gamma \circ \Phi$ where $\Phi : [c, d] \to [a, b]$ is a strictly increasing homeomorphism, i.e., orientation preserving. Then

$$\int_\Gamma f(z)dz = \int_c^d f(\Gamma(u))\Gamma'(u)du.$$

By chain rule, this is same as

$$\int_c^d f(\gamma \circ \Phi(u))\gamma'(\Phi(u))\Phi'(u)du.$$

Setting $t = \Phi(u)$ and applying change of variables, we see that the above is same as

$$\int_a^b f(\gamma(t))\gamma'(t)dt = \int_\gamma f(z)dz.$$

If Φ were orientation reversing then $\int_\gamma f(z)dz = -\int_\Gamma f(z)dz$.

Now suppose γ is a path. Then, we can write $\gamma = \gamma_1 + \cdots + \gamma_n$ where γ_i's are all $\mathcal{C}^1$ curves such that the ending point of γ_i coincides with the starting point of γ_{i+1}. Then we define

$$\int_\gamma f(z)dz := \sum_i \int_{\gamma_i} f(z)dz.$$

We have the following extremely useful inequality:

$$\left|\int_\gamma f(z)dz\right| \leqslant \int_a^b |f(\gamma(t))||\gamma'(t)|dt = \int_\gamma |f(z)|dz.$$

The use of this inequality is easily seen when f is bounded above by M. Then, we immediately see that

$$\left|\int_\gamma f(z)dz\right| \leqslant ML(\gamma)$$

The above inequality can be used to show that the index of a closed path with respect to a point on the unbounded component is 0:

$$\mathrm{Ind}(\gamma, w) = \frac{1}{2\pi} \int_\gamma \frac{dz}{z - w}$$

and therefore

$$|\mathrm{Ind}(\gamma, z)| \leqslant \frac{L(\gamma)}{2\pi d(w, \gamma^*)}.$$

The RHS clearly goes to 0 as $w \to \infty$ and we are done.

2 Vector fields

We want to study complex line integrals from the viewpoint of vector calculus. Complex line integrals are special cases of circulations of vector fields along curves. We will now study this in detail.

Definition 2. Let $U \subset \mathbb{R}^n$ be a domain. A *continuous vector field* on U is a map $\overrightarrow{X} : U \to \mathbb{R}^n$.

We can visualize a vector field as specifying vectors on each point of U starting at the point. The most common example of a vector field is the electric field or magnetic fields which were introduced by Faraday.

Example 3. The identity function on U is a vector field and it is a radial vector field.

Given a vector field $\overrightarrow{X}$ on U, we can define some special curves that "move" along the field. We say a $\mathcal{C}^1$ curve $\gamma : [a, b] \to \mathbb{C}$ is an integral curve or an orbit of $\overrightarrow{X}$, if

$$\gamma'(t) = \overrightarrow{X}(\gamma(t)) \quad \forall t \in [a, b].$$

Let $\overrightarrow{X}$ be a vector field on U. The standard example to keep in mind is the electric field or in general a force field. Let $\gamma : [a, b] \to U$ be a regular curve. Recall from physics that the work done to move unit mass along this path γ from $\gamma(a)$ to $\gamma(b)$ is given by

$$\int_\gamma \langle \overrightarrow{X}, \widehat{T} \rangle ds,$$

where $\widehat{T}(x)$ is the unit tangent vector to γ at x. Note that at $x = \gamma(t)$, $\widehat{T} = \frac{|\gamma'(t)|}{|\gamma'(t)|}$. Expanding the integral, we see that

$$\int_\gamma \langle \overrightarrow{X}, \widehat{T} \rangle ds = \int_a^b \langle \overrightarrow{X}(\gamma(t)), \frac{\gamma'(t)}{|\gamma'(t)|} \rangle |\gamma'(t)| dt = \int_a^b \langle \overrightarrow{X}(\gamma'(t)) \rangle dt.$$

We take the above as the definition of *circulation of* $\overrightarrow{X}$ *along the curve* γ. In terms of components, we see that the circulation is

$$\int_a^b X_1(\gamma(t)) x_1'(t) dt + \cdots + X_n(\gamma(t)) x_n'(t) dt,$$

which we abbreviate as

$$\int_\gamma X_1 dx_1 + X_2 dx_2 + \ldots X_n dx_n.$$

The integrated above is known as a continuous 1-form and we use the notation $\omega_{\overrightarrow{X}}. They are just a different wa$
then we define the 1-form

$$dh = \frac{\partial h}{\partial x_1} dx_1 + \cdots + \frac{\partial h}{\partial x_n} dx_n,$$

which is a notation familiar from calculus. Note that the vector field corresponding to dh is nothing but the gradient vector field of h.

Now, we relate the complex line integral of a function f on γ^* and circulation. We compute

$$\begin{aligned}\int_\gamma f(z)dz &= \int_a^b f(\gamma(t))\gamma'(t)dt \\ &= \int_a^b (u(x(t),y(t)+iv(x(t),y(t))(x'(t)+y'(t))dt \\ &= \int_a^b (ux'-vy')dt + i\int_a^b (uy'+vx')dt.\end{aligned}$$

The complex line integral of any complex-valued function has the circulation of $\bar{f}$ along γ as real part and circulation of $i\bar{f}$ along γ as imaginary part. This last observation motivates us to consider more general vector fields and forms:

$$\omega = P(x,y)dx + Q(x,y)dy,$$

where P and Q are now complex-valued functions. It is easy to see that under the more general notion of 1-form, the complex line integral of f along γ is same as the integral of the form $f(z)dx + if(z)dy$ along γ.

We now present a n-dimensional version of the fundamental theorem of calculus.

Theorem 4. *Let* $UI \subset \mathbb{R}^n$ *be open and* $\gamma : [a,b] \to U$ *be a path with starting point* A *and ending point* B. *Let* $h \in \mathcal{C}^1(U)$ *(real or complex-valued function) and let* $\overrightarrow{X} = dh$ *be the gradient vector field. Then*

$$\int_\gamma dh = h(B) - h(a).$$

In other words, the integral of a gradient vector field depends only on the endpoints and not on the path.

Proof. Let $x_i(t)$ be the components of $\gamma(t)$. Then by definition

$$\int_\gamma dh = \int_a^b \sum_i \frac{\partial h}{\partial x_i}(\gamma(t)x_i'(t)dt = \int_a^b \frac{d}{dt}(h\circ\gamma)(t)dt = h(B) - h(A).$$

□

Definition 5. A continuous vector field $\overrightarrow{X}$ on a domain $U \subset \mathbb{R}^n$ is said to be *conservative* if the circulation of $\overrightarrow{X}$ along any path depends only on the endpoints of the path. More precisely,

$$\int_{\gamma_1} \omega_{\overrightarrow{X}} = \int_{\gamma_2} \omega_{\overrightarrow{X}}.$$

Gradient vector fields are obviously conservative. A necessary and sufficient condition for a vector field to be conservative is the vanishing of the circulation along any closed path.

Theorem 6. *A continuous vector field* $\overrightarrow{X}$ *on a domain* $U \subset \mathbb{R}^n$ *is conservative iff it is a gradient vector field.*

Proof. We must show that every conservative vector field is the gradient vector field of some $h \in \mathcal{C}^1(U)$. The idea behind the proof is reminiscent of the proof that the existence of a branch of square root implies the existence of a branch of arg. Fix a point $x_0 \in U$ and define

$$h(x) := \int_{\gamma_x} \omega_{\overrightarrow{X}},$$

where γ_x is some path from x_0 to x. This is well-defined because, the vector field $\overrightarrow{X}$ is conservative. If $\overrightarrow{X} = (X_1, \dots, X_n)$, we must check that $\frac{\partial h}{\partial x_i} = X_i$. Now,

$$\frac{\partial h}{\partial x_i} = \lim_{t \to 0} \frac{h(x + te_i) - h(x)}{t} = X_i(x),$$

where $(e_1, \dots, e_n)$ is the standard basis of $\mathbb{R}^n$. Now, the value of $h(x+te_i)$ might be computed by integrating $\omega_{\overrightarrow{X}}$ along any path from x_0 to $x + te_i$; we take the path as the sum of two paths, one path from x_0 to x and then a straight line from x to $x + te_i$ which we parametrize as $x + ste_i, s \in [0, 1]$. Hence, the difference quotient just becomes

$$\int_0^1 X_i(x + ste_i)ds$$

and it is clear that as $t \to 0$, we have the limit $X_i(x)$ as required.

□

The function h which satisfies $dh = \omega_{\overrightarrow{X}}$ is said to be a *potential function* and the form $\omega_{\overrightarrow{X}}$ is said to be an *exact form*. It follows that a 1-form is exact iff its integral along any closed path is 0.

To check that a vector field $\overrightarrow{X}$ is conservative, one need not check that the circulation is zero over all paths. The following result reduces our effort if U is a star-like domain.

Theorem 7. *Let $U \subset \mathbb{R}^n$ be star-like. A vector field $\overrightarrow{X}$ is conservative iff its circulation is zero along the boundary of any triangle.*

Proof. In the proof of the previous theorem, fix x_0 to be the special with respect to which U is star-like and consider the path γ_x to be straight line-segments. □

Definition 8. We say that the form ω is *locally exact* on U if for each $x \in U$, we can find a small ball $B(x, r) \subset U$ such that for some $h \in \mathcal{C}^1(B(x, r))$, we have $\omega = dh$ on $B(x, r)$.

The following corollary is obvious:

Corollary 9. *A 1-form ω is locally exact on U iff only if its integral along the boundary of any triangle is 0.*

3 The fundamental theorem of complex calculus

Now, we introduce holomorphicity in the picture. Note that the complex line integral of the function f along a path $\gamma$is same as the integral of the form fdz along γ. Recall that

$$\frac{\partial f}{\partial z}(a) := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$
$$\frac{\partial f}{\partial \bar{z}}(a) := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

We also write $dz = dx + idy$ and $d\bar{z} = dx - idy$. It immediately follows that for a $\mathcal{C}^1$ function h (real or complex-valued)

$$dh = \frac{\partial h}{\partial z}dz + \frac{\partial h}{\partial \bar{z}}d\bar{z}.$$

Moreover, any 1-form $Pdx+Qdy$ might be written in the form $Adz+Bd\bar{z}$ where $A = 1/2(P-iQ)$ and $B = 1/2(P + iQ)$. Now if the form $f(z)dz$ is exact on the domain $U \subset \mathbb{C}$ then this means that there is a function $h \in \mathcal{C}^1(U)$ with the property that

$$dh = \frac{\partial h}{\partial z}dz + \frac{\partial h}{\partial \bar{z}}d\bar{z} = fdz.$$

This means that $\frac{\partial h}{\partial \bar{z}} = 0$ or in other words $h \in H(U)$. This shows that a form fdz is exact iff it has a holomorphic antiderivative.

Theorem 10. *Let f be a continuous function on the domain $U \subset \mathbb{C}$. Then the complex line integral of f along curves $\gamma : [a, b] \to U$ is independent of path iff f has a holomorphic anitderivative F on U. In this case one has:*

$$\int_\gamma f(z)dz = \int_\gamma F'(z)dz = F(B) - F(A).$$

Proof. Fix $z_0 \in U$ and for each z let γ_z be any path from z_0 to z and set

$$F(z) = \int_{\gamma_z} f(w)dw.$$

Again, $F(z)$ is well-defined. To prove $F'(z) = f(z)$, the difference quotient is $\frac{F(z+h)-F(z)}{h}$ which we can again evaluate by considering a path from z_0 to z and then considering the straight line path σ from z to $z + h$. Thus

$$F'(z) = \lim_{h\to 0} \frac{1}{h}\int_\sigma (f(w) - f(z))dw$$

and by ML-inequality the above integral is bounded by

$$\frac{1}{h} * \sup\{|f(w) - f(z)| : w \in \sigma^*\} * h \to 0$$

as $h \to 0$ as f is continuous.

Conversely, if $F'(z) = f(z)$ then

$$\int_\gamma f(z)dz = \int_a^b F'(\gamma(t))\gamma'(t)dt = \int_a^b (F\circ\gamma)'(t)dt = F(b) - F(a).$$

□

Any continuous function on $\mathbb{R}$ has an anti-derivative but for complex functions one requires the additional hypothesis of path independence of line integrals. The reason why we require this is because on the real-line there is only one “natural” path to go from a to b whereas on the complex plane there are many.

4 Examples

Most examples of vector fields come from physics. We give a few of them:

- Two important vector fields on $\mathbb{C}$ are given by the functions z and $-z$. The vector field z points radially outward whereas the vector field $-z$ points radially inward.
- The electric, magnetic and gravitational vector fields in space.
- Vector field for the movement of air on Earth will associate for every point on the surface of the Earth a vector with the wind speed and direction for that point. This can be drawn using arrows to represent the wind; the length (magnitude) of the arrow will be an indication of the wind speed. A ‘’high” on the usual barometric pressure map would then act as a source (arrows pointing away), and a ‘’low” would be a sink (arrows pointing towards), since air tends to move from high pressure areas to low pressure areas.
- Velocity field of a moving fluid. In this case, a velocity vector is associated to each point in the fluid.

Examples of complex line integrals

- Let $\gamma(t) = e^{it}, t \in [0, 2\pi]$ be the standard parametrization of the unit circle. Computing
$$\int_\gamma 1/z dz = \int_0^{2\pi} \frac{ie^{it}}{e^it} = 2\pi i.$$
This just means that the $\mathrm{Ind}(\gamma, 0) = 1$. On the other hand,
$$\int_\gamma 1/z|dz| = \int_a^b \frac{dt}{e^it} = 0.$$
- Let $f(z) = z^2$ and $\gamma = \gamma_1 + \gamma_2$, where γ_1 is the portion of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ and γ_2 is the vertical line segment from $(1,1)$ to $(1,0)$. We can parametrize γ_1 as $t + it^2, t \in [0,1]$ and γ_2 as the negative of the curve $1 + ti, t \in [0,1]$ and we see that
$$\int_{\gamma_1} z^2 dz = \int_0^1 (t + it^2)^2(1 + 2it) = \int_0^1 (t^2 - 5t^4 + 2it^3(2 - t^2))dt = 2/3(i - 1)$$
and
$$-\int_{\gamma_2} z^2 dz = -\int_0^1 (1 + ti)^2 i = -\int_0^1 i - 2ti - it^2 dt = 1 - 2/3i.$$
Hence the required line integral is $1/3$. We can evaluate this integral in much easier way by observing that the function z^2 has an anti-derivative and hence its complex-line integral is path independent. So, we may consider the path from 0 to 1 as the line segment and by the fundamental theorem of calculus for complex line integrals
$$\int_{\gamma_1+\gamma_2} z^2 dz = z^3/3|_1 + z^3/3|_0 = 1/3.$$

- Consider the function $f(z) = 1/m(z-a)^m$. This function has an derivative $(z-a)^{m-1}$. This shows that
$$\int_\gamma (z-a)^k dz = 0,$$
whenever $k \in \mathbb{N}$ along any closed path γ.
If $m < 0$ then F is holomorphic on $\mathbb{C} \setminus \{a\}$ and $F'(z) = (z-a)^{m-1}$. Hence if $k \neq -1$ is a negative integer and $a \notin \gamma^*$, again $\int_\gamma (z-a)^k dz = 0$.
If $k = -1$, then we already know that $\int_\gamma \frac{dz}{(z-a)} = 2\pi i \mathrm{Ind}(\gamma, a)$ which might not be 0. This shows that the function $\frac{1}{z-a}$ does not have an anti-derivative on $\mathbb{C} \setminus \{a\}$.
- Let us compute $\int_\gamma \sin z dz$ where γ is the arc of the curve $y = x^3$ from $(0, 0$ to $(1, 1)$. As $\sin z$ has anti-derivative $-\cos z$ the value of the integral is same as $-\cos(1+i) + \cos 0 = 1 - \cos(1+i)$.
- The standard methods of integration such as integration by parts and change of variables can be put to use when we are integrating functions that have anti-derivatives. Let $f, g \in H(U)$ and $F = f \cdot g$. Then $F' = fg' + gf'$ and consequently
$$F(B) - F(A) = \int_\gamma F'(z)dz = \int_\gamma f'(z)g(z)dz + \int_\gamma f(z)g'(z)dz.$$
Hence to evaluate $\int_\gamma z \sin z$ where γ is a path joining A and B and note that if $F = -z \cos z$ then $F'(z) = -\cos z + z \sin z$ and consequently
$$-B\cos B + A\cos A = \int_\gamma z \sin z dz + \int_\gamma -\cos z dz,$$
and hence the required integral is
$$\sin B - \sin A + A\cos A - B\cos B.$$

5 Domains with regular boundary

Definition 11. A domain $U \subset \mathbb{C}$ is said to be a *domain with regular boundary* if ∂U is a finite disjoint union of regular Jordan loops.

Similarly, we can define a domain with piece-wise regular boundary. Now, if U is a domain with regular boundary, then $\partial U = \gamma_1^* \sqcup \gamma_2^* \sqcup \cdots \sqcup \gamma_n^*$. We want to orient the boundary curves γ_i. Before doing this, we first need the following proposition which completely describes how such domains look like.

Proposition 12. *Let $U \subset \mathbb{C}$ be a domain with piece-wise regular boundary and let $\partial U = \gamma_1^* \sqcup \gamma_2^* \sqcup \cdots \sqcup \gamma_n^*$, where γ_i are regular Jordan loops. Then, after reindexing if necessary,*

1. $\gamma_1^* \subset \mathrm{Int}(\gamma_i), i = 2, \ldots, n$
2. $\mathrm{Int}(\gamma_2), \mathrm{Int}(\gamma_2), \ldots, \mathrm{Int}(\gamma_n)$ *are disjoint.*
3. $U = \mathrm{Int}(\gamma_1) \setminus (\overline{\mathrm{Int}}(\gamma_1) \cup \cdots \cup \overline{\mathrm{Int}}(\gamma_n))$.

Proof. The domain U is connected and therefore, it follows that that U is fully contained in either $\mathrm{Int}(\gamma_i)$ or $\mathrm{Ext}(\gamma_i)$. Our claim is that there is precisely one i, which we reindex as 1, such that $U \subset \mathrm{Int}(\gamma_i)$ and for $j \neq i$, we have

$$U = \mathrm{Int}(\gamma_1) \setminus (\mathrm{Int}(\gamma_2) \cup \ldots \mathrm{Int}(\gamma_n)) .$$

This claim will follow from the following observations:

- $\gamma_i^* \subset \mathrm{Int}(\gamma_j) \implies \mathrm{Int}(\gamma_i) \subset \mathrm{Int}(\gamma_j)$. To see this mote that $\overline{\mathrm{Ext}}(\gamma_j) \cap \gamma_i^* = \emptyset$. This means that the connected component $\mathrm{Ext}(gamma_j)$ must be fully contained in one of the connected components $\mathrm{Ext}(\gamma_i)$ or $\mathrm{Int}(\gamma_i)$. It cannot be $\mathrm{Int}(\gamma_i)$ as $\mathrm{Ext}(\gamma_j)$ is unbounded. Taking complements, we see that $\mathrm{Int}(\gamma_i) \subset \mathrm{Int}(\gamma_j)$.
- If $\mathrm{Int}(\gamma_i) \cap \mathrm{Int}(\gamma_j) = \emptyset$ then $U \subset \mathrm{Ext}(\gamma_i) \cap \mathrm{Ext}(\gamma_j)$. If $U \subset \mathrm{Int}(\gamma_i)$, then $\overline{U} \subset \mathrm{Int}(\gamma_i) \cup \gamma_i^*$ in which event it is not possible that $\gamma_j^* \subset \partial U$.

□

Definition 13 (Orientation). Let U be a domain with piece-wise regular boundary and let $\partial U = \gamma_1^* \sqcup \gamma_2^* \sqcup \cdots \sqcup \gamma_n^*$. We say that ∂U is positively oriented if on traversing along each γ_i in the direction of increasing t, the domain U is always to the left.

6 Green's formula

The following is the main result from multivariable calculus that we need to prove Cauchy's theorem.

Theorem 14 (Green's Theorem). *Let $U \subset \mathbb{C}$ be a bounded domain with positively oriented piecewise regular boundary and let $\partial U = \gamma_1^* \sqcup \gamma_2^* \sqcup \cdots \sqcup \gamma_n^*$. Let $\overrightarrow{X} = (P, Q)$ be a continuous vector field defined on a neighbourhood of $\overline{U}$ such that P, Q are differentiable functions and $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is continuous on $\overline{U}$. Then the following identity holds:*

$$\sum_i circ_{\gamma_i}(\overrightarrow{X}) = \sum_i \int_{\gamma_i} P dx + Q dy = \int\int_U \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy.$$

Remark 15. The integral over all the various curves γ_i is often denoted

$$\int_{\partial U} P dx + Q dy,$$

for obvious reasons. The proof is divided into several steps. First we will treat the special case, $U = [a, b] \times [c, d]$. If $P, Q \in \mathcal{C}^1(U)$ then this case is an easy consequence of the fundamental theorem of calculus and Fubini's theorem. To see this note that

$$\begin{aligned}
&\int_{\partial U}(P dx + Q dy)\\
&\quad = \int_a^b P(x,c)dx + \int_c^d Q(b,y)dy - \int_a^b P(x,d)dx - \int_c^d Q(a,y)dy\\
&\quad = \int_a^b (P(x,c) - P(x,d))dx + \int_c^d (Q(b,y) - Q(a,y))dy\\
&\quad = -\int_a^b \int_c^d \frac{\partial P}{\partial y} dxdy + \int_c^d \int_a^b \frac{\partial Q}{\partial x}(x,y)dxdy\\
&\quad = \int\int_U \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.
\end{aligned}$$

We cannot use the fundamental theorem of calculus in the proof of the general case.

Proof of Green's theorem.

Step 1. $\overline{U} = [a,b] \times [c,d]$. We use the notation $P_x := \frac{\partial P}{\partial x}$ and $P_y := \frac{\partial P}{\partial x}$ and similarly for Q. We also set

$$I(R) := \int_{\partial U}(P dx + Q dy) - \iint_U (Q_x - P_y)dxdy,$$

whenever R is a rectangle. We set $I := I(U)$. We divide U into four equal rectangle $U_i, i = 1,2,3,4$ and give them all the positive orientation. Hence,

$$I = \sum_{i=1}^4 I(U_i).$$

If $I \neq 0$, then, without loss of generality, we may assume that

$$|I| \leqslant 4|I(U_1)|.$$

We now set $I_1 = I(U_2)$ and repeat the process and so we can find a rectangles $\overline{U} \supset \overline{U}_1 \supset \overline{U}_2 \supset \dots \overline{U}_n \dots$ each obtained from the previous one by subdivision into four pieces and

$$|I| \leqslant 4^n |I(U_n)|.$$

Let z_0 be the intersection of the nested sequence of compact sets. Let $\varepsilon > 0$. Using Taylor's theorem, we can write

$$\begin{aligned}
P(x,y) &= P(z_0) + P_x(z_0)(x - x_0) + P_y(z_0)(y - y_0) + R_1(z)\\
Q(x,y) &= Q(z_0) + Q_x(z_0)(x - x_0) + Q_y(z_0)(y - y_0) + R_2(z),
\end{aligned}$$

where both $|R_1(z)|, |R_2(z)| \leqslant \varepsilon|z - z_0|$ if $|z - z_0| < \delta$. By the continuity of $Q_x - P_y$, we can assume that

$$(Q_x - P_y)(z) = (Q_x - P_y)(z_0) + R_3(z),$$

where $|R_3(z)| \leqslant \varepsilon$ if $|z-z_0| < \delta$. For n large, we must have $\overline{U}_n \subset D(z_0,\delta)$. Suppose $\overline{U}_n = [a_n,b_n]\times[c_n,d_n]$, one has

$$\begin{aligned}
&\int_{\partial U_n}(Pdx+Qdy)\\
&\quad= \int_{a_n}^{b_n} P(x,c_n)dx + \int_{c_n}^{d_n} Q(b_n,y)dy - \int_{a_n}^{b_n} P(x,d_n)dx - \int_{c_n}^{d_n} Q(a_n,y)dy\\
&\quad= P_y(z_0)(c_n-y_0)(b_n-a_n) + Q_x(z_0)(b_n-x_0)(d_n-c_n)\\
&\qquad - P_y(z_0)(d_n-y_n)(b_n-a_n) - Q_x(z_0)(a_n-x_0)(d_n-c_n) + R_n\\
&\quad= (b_n-a_n)(d_n-c_n)(Q_x(z_0)-P_y(z_0)) + R_n,
\end{aligned}$$

where

$$R_n = \int_{a_n}^{b_n}[R_1(x,c_n)-R_1(x,d_n)]dx + \int_{c_n}^{d_n}[R_2(b_n,y)-R_2(a_n,y)]dy.$$

For $z\in\partial U_n$, $|z-z_0|\leqslant L_n$, where L_n is the length of the diagonal of U_n. This shows that for n sufficiently large,

$$|R_n| < \varepsilon L_n P_n,$$

where P_n is the perimeter of U_n. We have $L_n = 2^{-n}L$ and $P_n = 2^{-n}P$ which means

$$|R_n| leq \varepsilon 4^{-n}\varepsilon LP.$$

We also have:

$$\iint_{U_n}(Q_x-P_y)dxdy = (Q_x-P_y)(z_0)(b_n-a_n)(d_n-c_n)+\widetilde{R}_n,$$

where $|\widetilde{R}_n|\leqslant \varepsilon 4^{-n}A$, A being the area of U. Hence

$$|I_n| \leqslant \varepsilon 4^{-n}(LP+A)$$

and thus

$$|I| \leqslant \varepsilon(LP+A)\to 0 \text{as } \varepsilon\to 0,$$

and this completes the proof of the first step.

Step 2. In this step we assume that the domain U is the region under the graph of a $\mathcal{C}^1$ function, i.e,

$$U = \{(x,y): a\leqslant x\leqslant b, 0\leqslant y\leqslant \phi(x)\},$$

where $\phi:[a,b]\to\mathbb{R}$ is $smoo^1$ smooth. We consider a partition of $[a,b]$, $a=x_0<x_1<\cdots<x_n=b$ formed by $n+1$ equidistant points and let $m_i = \inf\{\phi(x): x_{i-1}\leqslant x\leqslant x_i\} = \phi(t_i), t_i\in[x_{i-1},x_i]$. Let R_n be the union of the rectangles

$$\{(x,y): x_{i-1}\leqslant x\leqslant x_i, 0\leqslant y\leqslant m_i\}, i=1,\ldots,n.$$

By definition, each $R_n\subset\overline{U}$ and

$$\iint_{R_n}(Q_x-P_y)dxdy \to \iint_U (Q_x-P_y)dxdy$$

From the first step

$$\iint_{R^n}(Q_x-P_y)dxdy = \int_{\partial\mathbb{R}_n} Pdx+Qdy$$

Step 3. To be done.

7 Cauchy's theorem

Theorem 16. *Let $U \subset \mathbb{C}^n$ be a bounded domain with piece-wise regular boundary given positive orientation and let f be holomorphic in a neighborhood of U. Then*

$$\int_{\partial U} f dz = 0.$$

Proof. Follows from Green's theorem and the CR-equations. □

Applications.

Cauchy's theorem is very helpful in evaluating certain real integrals. The idea is extend the given integral on an interval to some complex line integral on some Jordan curve and then applying Cauchy's theorem.

1. Let us compute $\int_0 \to \infty \cos(t^2)dt$ and $\int_0^\infty \sin(t^2)dt$, i.e., $\lim_{R\to\infty} \cos(t^2)$ and $\lim_{R\to\infty} \sin(t^2)$. Consider the function $f(z) = e^{iz^2}$ integrated on the piece-wise regular path given by $\Gamma_1 = [0, R]$ followed by the arc of the circle of radius R that subtends an angle of $\pi/4$ at 0, call it Γ_2, and Γ_3 be the straight line segment from 0 to the endpoint of Γ_2. By Cauchy's theorem,

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz - \int_{\gamma_3} f(z)dz,$$

or in other words

$$\begin{aligned}\int_0^R \cos t^2 dt + i\int_0^R \sin t^2 dt &= \int_0^R e^{i(te^{i}\pi/4)^2}dt - \int_0^{\pi/4} e^{iR^2e^{2it}}Rie^{it}dt \\ &= \int_)^R e^{-t^2}e^{ipi/4}dt - \int_0^{\pi/4} e^{iR^2e^{2it}}Rie^{it}dt.\end{aligned}$$

Taking $R \to \infty$, we see that the first integral converges to $\sqrt{\pi}e^{i\pi/4}$. We estimate the second integral as follows

$$\begin{aligned}\left|\int_0^{\pi/4} e^{iR^2e^{2it}}Rie^{it}dt\right| &\leqslant R\int_0^{\pi/4} |e^{iR^2e^{2it}}|dt = R\int_0^{\pi/4} e^{-R^2\sin 2t}dt \\ &\leqslant R\int_0^{\pi/4} e^{-R^2t}dt = \frac{1}{R}(1 - e^{-\pi/4R^2}) < \frac{1}{R} \to 0.\end{aligned}$$

8 Existence of Logarithms

We now discuss the relationship between the existence of anti-derivatives and logarithms.

Proposition 17. *Let $U \subset \mathbb{C}$ be a domain and let $f \in H(U)$ without zeroes. Then there is a continuous branch of the logarithm of f iff the function f'/f has an anti-derivative on U.*

Proof. If let h be a branch of $\log f$. Let $z \in U$ and let D be a small disk around $f(z)$ that misses 0. Then as any two branches of the logarithm of f must differ on D by a constant integer multiple of 2π, it follows that we can find a continuous branch of $\log z$ on D such that $h = \log f$ on D. The chain rule gives that $h' = f'/f$.

Conversely, assume that f'/f has an anti-derivative h. Then the function $F = e^{-h}f$ satisfies

$$F'(z) = -e^{-h}h'f + e^{-h}f' = 0.$$

This means that $F = c \neq 0$ (a constant). Consequently, if $c = e^{\alpha}$, then $f = e^{h+\alpha}$ and $h + \alpha$ is a branch of the logarithm of f. $\square$

We now state a couple of simple corollary's of Cauchy's theorem pertaining- to the existence of logarithms.

Corollary 18. *Let $f \in H(U)$ then f locally has a holomorphic antiderivative on U. If U is star-like, then f globally has an antiderviative.*

Corollary 19. *If $f \in H(U)$ without zeroes on U, then f locally has the branch of logarithm on U. If U is star-like, then f globally has a branch of the logarithm.*