Functions of a Complex Variable

In this chapter, we will study functions of a complex variables. The most interesting functions on the complex plane are those that are holomorphic. The most important holomorphic functions are the exponential and the logarithm function. Holomorphicity is the complex analogue of differentiability, but for various reasons, which we shall study in great detail, a function being holomorphic is far more restrictive than the function being differentiable.

The focal point of this chapter will be power series and functions that have local power series expansions, i.e., *analytic functions*. The reason for this is two-fold:

(i). The most important functions that arise in practice are analytic.
(ii). All holomorphic functions are analytic!

The second point above will be one of the central theorems we will prove in this course.

We make some notation remarks. We will be considering functions $f : U \to \mathbb{C}$, where $U \subset \mathbb{C}$ is an open set. We will usually write $f = u + iv$, where $u := \mathrm{Re} f$ and $v := \mathrm{Im} f$ are the real and imaginary parts of f. Note that f is continuous iff u and v are continuous.

We begin with the definition and basic properties of holomorphic functions. We then present the theory of power series and study various properties of analytic functions.

1 Holomorphic functions

1.1 Definitions and examples

Definition 1. We say that f is $\mathbb{C}$-*differentiable at the point* $a \in U$ if

$$\lim_{h\to a} \frac{f(a+h) - f(a)}{h} \to 0.$$

In the case, we denote the limiting value by $f'(a)$ and call it the *derivative at* a.

If f is $\mathbb{C}$-differentiable at each point of U, then we say that f is *holomorphic on* U. The set of holomorphic mappings on U is denoted $H(U)$.

A function that is holomorphic on the whole of $\mathbb{C}$ is called an *entire function*.

Remarks.

a) If f is differentiable at a then f is continuous at a.
b) If f and g are differentiable at a then so are $f \pm g, cf (c \in \mathbb{C})$ and fg and the usual rules of differentiation apply. The proofs of these facts follows *mutatis mutandis* from the proofs of the real case.
c) If f is differentiable at a and $f(a) \neq 0$, then $\frac{1}{f}$ is differentiable at a and its derivative at a is $\frac{-1}{f(a)^2}$.

d) We can rewrite the existence of the derivative a as follows: We say that if f is differentiable at a if there exists $\alpha \in \mathbb{C}$ such that
$$f(a+h) = f(a) + \alpha h + o(h). \tag{1.1}$$
Note that the above says that f can be approximated by an affine linear map near a with the error term being $o(h)$.

e) If we consider f as map from $U \subset \mathbb{R}^2$ to $\mathbb{R}^2$, then note that f being $\mathbb{R}$-differentiable at a is equivalent to the existence of an $\mathbb{R}$-linear map $df(a)$ such that
$$f(a+h) = f(a) + df(a)h + o(h), \quad h \in \mathbb{R}^2.$$
Combining this with 1.1 , we see that f is $\mathbb{C}$-differentiable iff f is $\mathbb{R}$-differentiable and the derivative map $df(a)$ is $\mathbb{C}$-linear.

f) The previous remark shows that the chain rule is valid for a composition of $\mathbb{C}$-differentiable functions. Please write down a precise statement and give a three line proof.

Examples.

(i) The function $f(z) = z^n$ is entire and its derivative at the point $a \in \mathbb{C}$ is given by na^{n-1}. The proof of this follows *mutatis mutandis* from the proof of the real case.

(ii) Combining the above with the preceding remarks, we see that any polynomial $P(z)$ is entire.

(iii) Consider the function $f(z) = \overline{z}$. In real coordinates this the map $(x, y) \to (x, -y)$ which is linear and hence this map is $\mathcal{C}^\infty$-smooth. However, for any $a \in \mathbb{C}$, note that
$$\frac{f(a+h) - f(h)}{h} = \frac{\overline{h}}{h}.$$
If $h \to 0$ along the real axis, then the above limit $\to 1$. On the other hand, if $h \to 0$ along the imaginary axis then the above limit $\to -1$ showing that f is *not* $\mathbb{C}$-differentiable at any point $a \in \mathbb{C}$.

(iv) Consider the function $f(z) = |z|^2 = z \cdot \overline{z}$. In real coordinates, we can consider at as the function $(x, y) \to x^2 + y^2$. This is polynomial function in x, y and hence $\mathcal{C}^\infty$-smooth. Again computing as before, we see that the difference quotient if of the form
$$\overline{a} + a\frac{\overline{h}}{h} + \overline{h}.$$
As $h \to 0$, the above difference quotient has a limit iff $a = 0$. Hence $|z|^2$ is $\mathbb{C}$-differentiable only at the point 0.

The last two examples show that f being $\mathbb{C}$-differentiable at a point a is a severe restriction. The limit of the difference quotient must exist and agree on all possible approaches to the point a. We could approach a along any line, or even complicated curves like spirals and the limiting value must exist and agree on any such approach. This is the crucial difference between real differentiability of a function defined on an interval and $\mathbb{C}$-differentiability. The difference between f being differentiable as a function on $\mathbb{R}^2$ and as a $\mathbb{C}$-differentiable function is that the map $df(a)$ is now $\mathbb{C}$-linear which means that it is conformal. This will force holomorphic functions to behave more nicely that $\mathcal{C}^\infty$ functions.

1.2 The Cauchy–Riemann equations

We would like some necessary and sufficient conditions for f to be $\mathbb{C}$-differentiable at a. The following is the famous Cauchy–Riemann equations. Let U, f, a be as before and write z as $x + iy, x, y \in \mathbb{R}$ and $f = u + iv$ where u and v are real-valued. Assume that a, the partial derivatives u_x, u_y, v_x, v_y exist. We define

$$\frac{\partial f}{\partial z}(a) := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$
$$\frac{\partial f}{\partial \bar{z}}(a) := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

Theorem 2. *Assume that f is $\mathbb{R}$-differentiable. Then the following conditions are all equivalent:*

(i) f *is $\mathbb{C}$-differentiable at* a.
(ii) $u_x = v_y$ *and* $u_y = -v_x$.
(iii) $\frac{\partial f}{\partial y}(a) = i\frac{\partial f}{\partial x}(a)$.
(iv) $\frac{\partial f}{\partial \bar{z}}(a) = 0$.

Proof. The equivalence of (ii), (iii) and (iv) are obvious. Let us see how the map df_a acts on the two elements $1, i \in \mathbb{C}$. Now $1, i$ are nothing but the vectors $(1, 0)$ and $(0, 1)$, respectively, in $\mathbb{R}^2$. This $df_a(1) = \frac{\partial f}{\partial x}(a)$ and $df_a(i)\frac{\partial f}{\partial y}(a)$. Now, df_a is $\mathbb{C}$-linear iff $df_a(i) = i df_a(1)$ and this proves the equivalence. □

Remarks.

a) Any of the equivalent equations $(ii) - (iv)$ is called the *Cauchy–Riemann* equations. Note that a function being $\mathbb{C}$-differentiable is equivalent to f being *both* $\mathbb{R}$-differentiable and satisfying the Cauchy–Riemann equation. Just one of the latter conditions being satisfied *does not* guarantee $\mathbb{C}$-differentiability. For instance, the function $f = \sqrt{|xy|}$ satisfies the Cauchy–Riemann equations at 0 but is not $\mathbb{C}$-differentiable at 0.
b) Note that the $df(a)$ written as a matrix (i.e., the Jacobian matrix) is
$$\begin{pmatrix} u_x & u_y \\ -v_x & v_y \end{pmatrix}.$$
The fact that this matrix is $\mathbb{C}$-linear forces $u_x = v_y$ and $u_y = -v_x$ which is another way of seeing that the Cauchy–Riemann equations are necessarily satisfied by a $\mathbb{C}$-differentiable map.
c) Yet an another way of seeing that the Cauchy–Riemann equations are satisfied by a $\mathbb{C}$-differentiable function is directly from definition. Let f be $\mathbb{C}$-differentiable at a, then
$$\lim_{h \to a} \frac{f(a+h) - f(a)}{h},$$
can be evaluated by taking h to 0 along the real axis, i.e., the above limit is nothing but $\frac{\partial f}{\partial x}$. Alternatively, we may take $h \to 0$ along the imaginary axis, in which case the above limit is $-i\frac{\partial f}{\partial y}$ which is nothing but another form of Cauchy–Riemann equations. This also shows that $f'(a) = u_x(a) + v_x(a)$

d) If f is $\mathbb{C}$-differentiable at a, then $\det(J_f(a)) = u_x^2(a) + v_x(a)^2 = |f'(a)|^2$.

The above remarks motivates the question: If f satisfies the Cauchy–Riemann equations at a then under what additional hypotheses on f does it follow that f is $\mathbb{C}$-differentiable at a? The literature on this topic is quite vast and we will be content by stating the following famous theorem.

Theorem 3 (Looman–Menchoff). *Suppose the partial derivatives of f exist on U and $\frac{\partial f}{\partial \bar{z}} \equiv 0$. Then $f \in H(U)$.*

We end this section with the following simple consequence of the Cauchy–Riemann equation.

Proposition 4. *Let f be a real-valued function and suppose that f is $\mathbb{C}$-differentiable at a. Then $df(a) = 0$. Consequently any real-valued holomorphic function on a domain U is constant. If $f, g \in H(U)$ that have the same real or imaginary parts, then $f - g \equiv c$ where $c \in \mathbb{C}$ is some constant.*

Proof. If $f = u + iv, v \equiv 0$ then the Cacuhy–Riemann equations show that $u_x(a) = v_y(a) = 0$ and consequently $f'(a) = \frac{\partial f}{\partial x} = u_x(a) + iv_x(a) = 0$. The other parts are easy. $\square$

1.3 Holomorphic functions and conformality

The fact that the derivative of a holomorphic mappings is $\mathbb{C}$-linear suggest that holomorphic mappings should be conformal in some appropriate sense. We make this precise in this section. First, we consider any arbitrary $\mathbb{R}$-differentiable mapping $f : U \to \mathbb{R}^n$ on an open set $U \subset \mathbb{R}^n$. Let I be a closed interval that has 0 as an interior point and let $\gamma : I \to U$ be a differentiable curve. Then $f \circ \gamma$ is also a differentiable curve and by the chain rule

$$\frac{d}{dt}|_{t=0}(f \circ \gamma)(t) \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\gamma(t)) \cdot \gamma_i'(0) = df(a)(\gamma'(0)).$$

This shows that the tangent of the curve γ at the point $a := \gamma(a)$ is transformed to the tangent vector of the curve $f \circ \gamma$ at $f(a)$ by the map $df(a)$.

If γ_1 and γ_2 are two differentiable curves that pass through the point a, we define the (oriented) angle between γ_1 and γ_2 at a to be the oriented angles between the tangent vector of γ_1 at a and the tangent vector of γ_2 at a.

Definition 5. We say that the map f is *conformal at* a if $df(a)$ is invertible and a conformal linear map.

Remarks.

(i) Note that his means that if γ_1 and γ_2 are regular curves that intersect at a then the angles between γ_1 and γ_2 at a is same as the angles between $f \circ \gamma_1$ and $f \circ \gamma_2$ at $f(a)$.

(ii) From the result5s of the previous chapter, f is conformal at a iff f is $\mathbb{C}$-differentiable at a and $f'(a) \neq 0$. In this event, if γ is a curve passing through a such that $\gamma(0) = a$ and $\gamma'(0) \neq 0$ then then tangent vector to $f \circ \gamma$ at $f(a)$ is give by $f'(a) \cdot \gamma'(0)$. As multiplication by a complex number is just a dilation followed by a rotation, it is clear why $\mathbb{C}$-differentiable mappings are conformal.

2 Power series

So far all the examples of holomorphic functions we have given were all polynomial functions. In this section, we develop the machinery of power series which will allow us to give more examples of holomorphic functions. In particular, we will be able to extend the definitions of the exponential and trigonometric functions to $\mathbb{C}$. We begin with a few remarks on polynomial and rational functions.

2.1 Absolutely convergent series

I will assume that you are familiar with the basic properties of a series of complex numbers. We briefly recall some of the deeper results on absolutely convergent series of complex numbers. For more detailed treatments please look at "Baby Rudin".

Definition 6. We say that a series of complex numbers $\sum_n c_n$ is *absolutely convergent* or *summable with sum* s if $\sum_n c_n \to s$ and $\sum_n |c_n|$ converges.

Proposition 7. *Let $(z_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers.*

a) The series $\sum_{n=0}^{\infty} |z_n|$ converges iff $\sum_{n=0}^{\infty} z_n$ converges with sum S and for any bijective mapping $\sigma : \mathbb{N} \to \mathbb{N}$, the series $\sum_{n=0}^{\infty} z_{\sigma(n)}$ converges also with sum S. In other words, a series of complex numbers is absolutely convergent iff the all rearrangements of the series converge to the same sum.

b) Suppose $Z^+ = \bigsqcup_{i\in I} E_i$, where I is a countable indexing set. If the series $\sum_{n=0}^{\infty} |z_n|$ converges then each of the series $\sum_{l\in E_i} z_l$ converges with sum S_i and furthermore the series $\sum_{i\in I} S_i$ converges with sum S, where the series $\sum_{n=0}^{\infty} z_n$ converges with sum S. In other words, any absolutely convergent series might be summed in arbitrary blocks and all such sums agree. This property is called associativity.

Proof. Let s_n denote the n-th partial sum of the series $\sum z_n$ and s'_n that of the series $\sum z_{\sigma(n)}$. Let $\varepsilon > 0$. First choose N_0 so large so that $\sum_{n=N_0}^{m} |z_n| < \varepsilon, m \geqslant N_0$ and then choose $N_1 \geqslant N_0$ so that $\sigma(\{1, 2 \ldots, N_1\}) \supset \{1, 2, \ldots, N_0\}$. Then it is clear that $|s_n - s'_n| < \varepsilon$ proving that the series $\sum z_{\sigma(n)}$ converges to the same sum as $\sum z_n$.

The converse of this theorem is the famous theorem of Riemann on rearrangements of series. We will sketch a prove of the result for a series of real numbers. Suppose that $\{z_n\}$ is conditionally convergent. First define $z_n^+ := \frac{z_n + |z_n|}{2}$ and $z_n^- := \frac{z_n - |z_n|}{2}$. The series $\sum z_n^+$ and $\sum z_n^+$ are nothing but the sums of the positive and negative terms, respectively. Note that both these series diverge (why?). Now, let $a > 0$. We will prove that we can find a rearrangement of $\{z_n\}$, say $\{y_n\}$ that converges to a. As $\sum z_n^+$ diverges, the partial sums will eventually exceed a. The first few terms of our rearranged series will be taken the initial terms of $\{z_n^+\}$ so that the partial sums just exceed a. Now, we add the initial terms from $\{z_n^-\}$ so that the sum of the rearranged series is now just $< a$. We then add terms from $\{z_n^+\}$ again so that the sum just exceeds a again. And now terms for $\{z_n^-\}\ldots$. We repeat this process. We can do this because both $\sum z_n^+$ and $\sum z_n^-$ diverge. It is easy to see that the rearranged series does in fact converge to a (this uses the fact that $z_n \to 0$). All the gory details of this argument is present in "Baby Rudin".

The second part is left as an exercise. $\square$

Example 8. We will now consider a double series $\sum_{n,m} z_{n,m}$. So we have a infinite matrix of complex numbers which we might sum in any way we please: row-wise, column-wise, along diagonals, etc. There is no guarantee that all these sums agree. For instance take $z_{n+1,n} = 1$ and $z_{n,n+1} = -1$ and all other entries 0. Summing by rows we get -1 and summing by columns we get 1. But if we could show that in some irder $\sum_{m,n} |z_{m,n}|$ is finite then by the previous result we can sum up the series in any way we please.

Example 9. For $q \in \mathbb{C}, |q| < 1$, we have

$$\sum_{m=1}^{\infty} \frac{mq^m}{1-q^m} = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2}.$$

To show this, we first expand

$$\frac{q^n}{1-q^m} = \sum_{m=1}^{\infty} q^{nm} \text{ and } \frac{q^n}{(1-q^m)^2} = \sum_{m=1}^{\infty} mq^{nm}.$$

We will now show that $(mq^{mn})_{n,m\geqslant 1}$ is summable in $\mathbb{C}$. Then the result follows from associativity. To do this, we take absolute values to pass to a series in $\overline{\mathbb{R}}_+$ where we can sum up in any way we choose:

$$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} m|q|^{mn} \right) = \sum_{n=1}^{\infty} \frac{|q|^m}{(1-|q|^n)} < \infty.$$

This finishes the proof.

Proposition 10. *Let* $\sum_n a_n$ *and* $\sum_n b_n$ *be summable with sums* A *and* B*. Then the series* $\sum c_k$*, where* $c_k = \sum_{n=0}^{k} a_n b_{n-k}$*, is summable with sum* AB*.*

Proof. The series $\sum c_k$ is summable as

$$\sum_k |c_k| \leqslant \sum_{m,n} |a_n||b_m| = (\sum_m |a_n|)(\sum_n |b_n|) < \infty.$$

This means that the series $\sum_{n,m} a_n b_m$ is summable. Summing first in m and then in n we get AB. Grouping the terms $a_{n,m}$ in blocks where $n + m = k$, we get $\sum c_k$. ☐

2.2 Sequences and series of functions

Throughout this section, X will be a set and $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of complex valued functions on E.

Definition 11. We say that f_n *converges uniformly to* $f : X \to \mathbb{C}$ if for each $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that

$$|f_n - f| < \varepsilon,$$

whenever $n \geqslant N$.

We say that the series $\sum_n f_n$ converges uniformly if the sequence of partial sums converge uniformly.

Remarks.

(i) We have a Cauchy criterion for uniform convergence: for each $\varepsilon > 0$, we must have:

$$|f_n(x) - f_m(x)| < \varepsilon,$$

for m, n sufficiently large, independent of the point $x \in X$.

(ii) If X were a metric space then the limit of any uniformly convergent sequence (or series) of functions on X is automatically continuous.

Proposition 12 (Weirstrass M-test). *If* $|f_n(x)| \leqslant M_n$ $\forall x \in X$ *and* $\sum M_n < \infty$ *then the series* $\sum f_n$ *is uniformly convergent.*

Proof. This follows in a straightforward manner from Cauchy's criterion. For $\varepsilon > 0$ and $m > n$, we have

$$\left|\sum_n^m f_k(p)\right| \leqslant \sum_m^n |f_k(p)| \leqslant \sum_{k=n}^{\infty} M_k \varepsilon,$$

for n sufficiently large. □

The following identity is due to Abel and will be needed in proving Abel's convergence theorem for power series.

Lemma 13. *Let* $(a_n), (b_n)$ *be two sequences of complex numbers and write* $A_n = a + \cdots + a_n$. *Then*

$$\sum_{n=1}^{k} a_k b_k = A_n b_{n+1} - \sum_{k=1}^{n} (A_k(b_{k+1} - b_k), \quad n \in \mathbb{N}.$$

Proof. Setting $A_0 = 0$, we see that

$$\begin{aligned}\sum_{k=1}^{n} a_k b_k &= \sum_{k=1}^{n} (A_k - A_{k-1}) b_k \\ &= \sum_{k=1}^{n} A_k b_k - \sum_{k=1}^{n-1} A_k b_{k+1} = \sum_{k=1}^{n} A_k (b_k - b_{k+1}) + A_n b_{n+1}.\end{aligned}$$

□

2.3 Power series

Definition 14. Let $\{a_n\}_n \in \mathbb{N}$ be a sequence of complex numbers. Then the *power series with coefficients* $\{c_n\}$ *centred at* a is the series of functions

$$\sum_{n=0}^{\infty} c_n (z - a)^n.$$

The *domain of convergence* of the power series is the set

$$E := \{z \in \mathbb{C} : \sum c_n (z - a)^n \text{ is convergent}\}.$$

We are interested in the set E and the properties of the limit function $f : E \to \mathbb{C}$. For instance:

1. Is the convergence of the power series uniform in E?
2. Is E open? Closed? Connected?

We use the notion of *upper limit* of a sequence of numbers.

Definition 15. Let (x_n) be a sequence of real numbers then the set of subsequential limits E is the set of all points x in $\overline{\mathbb{R}}$ (extended real numbers) such that some subsequence of (x_n) converges to x. We define the *upper limit* of (x_n) to be

$$\limsup_n x_n := \sup E.$$

If $\rho = \limsup_n x_n$ is finite then it is the unique number that satisfies the following two properties:

1. $\rho \in E$.
2. If $x > \rho$, only finitely many elements of the sequence (x_n) are $\geqslant x$.

Theorem 16. *Let $\sum_{n=0}^{\infty} c_n(z-a)^n$ be a power series. Let $R := \frac{1}{\rho}$, where $\rho = \limsup_n |c_n|^{1/n}$ (here $R := 0$ if $\rho = \infty$ and $R := \infty$ if $\rho = 0$). Then the series converges uniformly on compact subsets of the open unit disk $D(a, R)$ and diverges on $\mathbb{C} \setminus \overline{D}(a, R)$. Furthermore, for each $z \in D(a, R)$ the convergence is absolute. Thus $D(a, R) \subset E \subset \overline{D}(a, R)$.*

Proof. If $|z-a| > R$, then $|z-a|^{-1} < \rho$ and hence there are infinitely many terms $|c_n|^{1/n}$ that are greater than $|z-a|^{-1}$, i.e., infinitely many terms of the form $|c_n(z-a)^n| > 1$. Therefore, the sequence $(c_n(z-a)^n)$ cannot converge to 0 showing that $z \notin E$. On the other hand if $r < R$ and $|z-a| \leqslant r$ then $|c_n(z-a)^n| \leqslant |c_n| r^n =: M_n$ and $M_n^{1/n} = |c_n|^{1/n} r$. As $\limsup_n |c_n|^{1/n} = 1/R$, this means that given $\varepsilon > 0$, for large n, $|c_n|^{1/n} < \frac{1}{R} + \varepsilon$ and thus

$$|c_n|^{1/n} r < \frac{r}{R} + r\varepsilon < 1,$$

for ε small. Thus by the root test, $\sum M_n$ converges and Weirstrass's M-test delivers the proof. $\square$

The number R above is called the *radius of convergence* of the power series and the expression $\rho = \limsup_n |c_n|^{1/n}$ is called *Hadamard's formula.*

Example 17. Consider the geometric series $\sum_n z^n$. Then each $c_n = 1$ and hence $R = 1$ and the domain of convergence is the unit disk $\mathbb{D}$. The convergence is *not* uniform on $\mathbb{D}$. For otherwise, the limit $\frac{1}{1-z}$ would have to be bounded. Also note that the series $\sum_n z^{2m}$ has coefficients $c_n = 1$ if n is even and 0 otherwise, has he same radius of convergence. It is easy to see that if the series $\sum_n c_n z^n$ has radius of convergence R then the series $\sum_n c_n w^{kn}$ obtained by substituting $z = w^k$ has radius of convergence $R^{1/k}$.

Also note that

$$\begin{aligned} R &= \sup\{r > 0 : \sum_n c_n(z-a)^n \text{ converges for } z \in D(a, r)\} \\ &= \sup\{r > 0 : \sum_n |c_n| r^n < \infty\} \end{aligned}$$

Given two series $\sum_n c_n(z-a)^n$ and $\sum_n d_n(z-a)^n$ whose radii of convergence are R_1 and R_2, respectively, we can form both the sum and Cauchy product of the two series. From the above relations it is clear that the radius of convergence of the sum is $\geqslant \min(R_1, R_2)$. As the Cauchy product of two summable series is also summable, it is clear that the radius of convergence of the Cauchy product also has radius of convergence $\geqslant \min(R_1, R_2)$.

Example 18. The series $\sum_{n\geqslant 1} \frac{z^n}{n^2}$ has radius of convergence 1 and converges uniformly on $\overline{\mathbb{D}}$ by Weirstrass M-test as $|\frac{z^n}{n^2} \leqslant \frac{1}{n^2}$.

In general, the domain of convergence will contain some points on the circle $C(a, R)$.

Example 19. The series $\sum_{n\geqslant 1} \frac{z^n}{n}$ has radius of convergence 1. If $z = 1$, the series divergence. On the other hand, when $z = -1$, it converges. We will say more about this series after study more convergence tests.

2.4 Analytic functions

Given a power series $\sum c_n(z-a)^n$, we can formally write down its derivative $\sum nc_n(z-a)^{n-1}$.

Proposition 20. *Let $a \in \mathbb{C}$. We define the power series $S(z) = \sum c_n(z-a)^n$ and the series $S'(z) = \sum nc_n(z-a)^{n-1}$. Then the radius of convergence of S and S' are equal.*

Proof. Let R be the radius of convergence of $S(z)$ and R' be the radius of convergence of $S'(z)$. Let $z \in \mathbb{C}$ be such that $|z-a| < R'$. Then by the root test

$$1 \geqslant \limsup |nc_n(z-a)^{n-1}|^{1/n} > \limsup |c_n(z-a)^n|^{1/n},$$

as $n|z-a|^{n-1} > |z-a|^n$ for $n > |z-a|$. From the root test, it follows that $S(z)$ also converges.

On the other hand, suppose that $S(z)$ converges absolutely. Choose z_0 such that $|z-a| < |z_0 - a| < R$. By the root test, it follows that $|c_n(z-a)^n|^{1/n} < 1$ for all sufficiently large n. Then:

$$\begin{aligned}
1 &= \limsup |n|^{1/n} \\
&> \limsup |n|^{1/n} \frac{|z-a|}{|z_0-a|} \\
&= \limsup \left| n \frac{|z-a|}{|z_0-a|}^{n-1} \right|^{1/n} \\
&\geqslant \left| n \frac{|z-a|}{|z_0-a|}^{n-1} c_n(z-a)^{n-1} \right|^{1/n} \\
&= \limsup |nc_n(z-a)^{n-1}|^{1/n}.
\end{aligned}$$

Hence, $R' \geqslant R$ and hence $R = R'$. □

Theorem 21. *If the power series $\sum nc_n(z-a)^{n-1}$ has radius of convergence $R > 0$, then the limit function $f(z) = \sum c_n(z-a)^n$ is holomorphic in the disk $D(a, R)$ and $f'(z) = \sum nc_n(z-a)^{n-1}$.*

Proof. Fix $z \in D(a, R)$ and let $\varepsilon > 0$. We have to show that

$$\left| \frac{f(w) - f(z)}{w - z} - \sum_n n c_n (z - a)^{n-1} \right| < \varepsilon,$$

for w suitably close to z. We write $f(z)$ as $S_N(z) + R_N(z)$ where S_N is the n-th partial sum of the power series of f and R_N is the tail and break above the above formula as the sum of three terms $I + II + III$:

$$\left(\frac{S_N(w) - S_N(z)}{w - z} - \sum_{n=0}^{N} n c_n (z - a)^n \right) - \sum_{N}^{\infty} n c_n (z - a)^n + \frac{R_N(w) - R_N(z)}{w - z}.$$

The last term is

$$\sum_{n>N} c_n ((w - a)^{n-1} + w - a)^{n-2}(z - a) + \cdots + (w - a)(z - a)^{n-2} + (z - a)^{n-1}),$$

where we just expanded out $((w - a) - (z - a))^{n-1}$. Suppose $|z - a| < \rho < R$ and $|w - a| < \rho$, then $|III| \leqslant \sum_{n>N} n|c_n|\rho^{n-1}$. Since $\rho < R$, this is the tail of a a convergent series, so we may choose N so large so that $|III| < \varepsilon$ (here it is irrelevant whether z is close to w or not). Similarly, we can assume that N is so large that $|II| < \varepsilon$ as well as II is also the tail of a convergent series. Finally, it clear that for w sufficiently close to z, $|I| < \varepsilon$. This because S_N is a polynomial whose derivative is $\sum_{n=0}^{\infty} n c_n (z - a)^n$. Hence, we are done. $\square$

Applying the above result iteratively, we see that f is infinitely-differentiable and the n-th derivative f is given by a power series expansion that converges in the disk $D(a, R)$.

We might, of course, differentiate a power series term-by-term to obtain its derivative. Repeating this process n-times, the first term of the new series is $n!c_n$. Substituting $z = a$ in this new series gives $f^n(a)$. We see that $c_n = \frac{f^n(a)}{n!}$. This shows that the if f has a power series expansion centred at a point a with radius of convergence $R > 0$, then the coefficients of this expansion are uniquely determined.

Now, suppose that f is a holomorphic function on $D(a, R)$ and suppose f' has a power series expansion

$$f'(z) = \sum_{n=0}^{\infty} d_n (z - a)^n,$$

that converges in $D(a, R)$ then f has the power series expansion

$$f(z) = f(a) + \sum_{n=0}^{\infty} \frac{d_n}{n+1} (z - a)^{n+1}.$$

Indeed, the RHS above is a convergent power series on $D(a, R)$ and, calling the holomorphic function it defines g, we see that $f' \equiv g'$ and $f(a) = g(a)$. Hence, the holomorphic function $f - g$ has a derivative that vanishes on $D(a, R)$ and hence $f \equiv g$.

Definition 22. Let $f : U \to \mathbb{C}$ be a function where $U \subset \mathbb{C}$. We say that f is *complex-analytic* if for each $a \in U$, we can find a power series centred at a with disk of convergence $D(a, R) \subset U$ whose values agree with those of f, i.e., f can be locally expressed as a power series.

The size of the disk of convergence will, in general, vary with the point z. However, we will later prove that it is at least $d(z, \partial U)$. In other words, the power series expansion of an analytic function at a point z defines the values of the function on the largest disk centred at that point wholly contained in U.

If $\sum_n c_n (z-a)^n$ is a power series convergent in $D(a, R)$, it is not at all obvious that it defines an analytic function in $D(a, R)$. This is the content of the following

Theorem 23. *Let* $f(z) = \sum c_n (z-a)^n$ *be a convergent power series in* $D(a, R), R > 0$ *and let* $b \in D(a, R)$. *Then the series* $\sum_{n=0}^{\infty} \frac{f^{(n)}(b)}{n!}(z-b)^n$ *has radius of convergence at least* $R - |a-b|$ *and is the power series expansion of* f *centred at* b.

Proof. As the derivatives of a power series can be obtained by differentiating term-by-term, we see that

$$f^n(b) = \sum_{m=0}^{\infty} \frac{(n+m)!}{m!} c_{n+m} (b-a)^m,$$

and hence

$$|f^{(n)}(b)| \leqslant \sum_{m=0}^{\infty} \frac{(n+m)!}{m!} |c_{n+m}| |b-a|^m.$$

Let r be such that $|b-a| \leqslant r < R$ and take a point $z \in D(b, R-r)$. We will show that the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(b)}{n!} (z-b)^n$$

is absolutely convergent. We compute

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{|f^{(n)}(b)|}{n!} |z-b|^n &\leqslant \sum_{n=0}^{\infty} \frac{|f^{(n)}(b)|}{n!} (r - |b-a|)^n \\
&\leqslant \sum_{n,m=0}^{\infty} \frac{(n+m)!}{m!n!} |c_{n+m}| |b-a|^m (r - |b-a|)^n \\
&= \sum_{k=0}^{\infty} |c_k| \sum_{m+n=k} \frac{(n+m)!}{m!n!} |b-a|^m (r-|b-a|)^n = \sum_{k=0}^{\infty} |c_k| r^k < \infty.
\end{aligned}$$

This shows that the series $\sum_{n=0}^{\infty} \frac{f^{(n)}(b)}{n!}(z-b)^n$ has radius of convergence at least $R - |b-a|$. The sum of the series can be computed by summing up arbitrarily in blocks and a computation similar to the one above shows that for a point $z \in D(b, R - |b-a|)$, we have

$$f(z) = \sum c_n (z-a)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(b)}{n!} (z-b)^n,$$

proving the result.

$\square$

3 Examples of Analytic functions

3.1 Polynomials and rational functions

The simplest class of functions on $\mathbb{C}$ are the polynomials:

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n, \quad a_j \in \mathbb{C} \text{ and } a_n \neq 0.$$

The number n is called the *degree of* P and denoted $\operatorname{def}(P)$. We will use the notation $\mathbb{K}[x]$ to denote the ring of all polynomials in the variable x over the field $\mathbb{K}$. Obviously, a polynomial $P : \mathbb{C} \to \mathbb{C}$ is a continuous function. We may write $P(z) = P_1(x, y) + iP_2(x, y)$, where $P_1, P_2 \in \mathbb{R}[x, y]$ are real-valued polynomials in the variables x and y. We can also substitute $z = x + iy$ in the expression of P to obtain a polynomial $P(x, y) \in \mathbb{C}[x, y]$. An interesting question: given a polynomial $P(x, y) = P_1(x, y) + iP_2(x, y), P_1, P_2 \in \mathbb{R}[x, y]$, under what conditions does it follow that $P \in \mathbb{C}[z]$? A detailed answer is given in the textbook. We will be content to give a simple answer. Set $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$ and substitute in the expression for P and check that after simplification, there are no terms involving $\bar{z}$.

A similar question is the following: if $P \in \mathbb{C}[z, \bar{z}]$, when does it follow that $P \in \mathbb{R}[x, y]$? Well, writing

$$P(z, \bar{z}) = \sum_{k+l \leqslant n} a_{k,l} z^k \bar{z}^l,$$

we see that if P is to take only real values, it follows that $\overline{P(z)} = P(z)$ and this can happen iff $a_{k,l} = \bar{a}_{l,k}$.

A polynomial is obviously a complex-analytic function on the whole of $\mathbb{C}$. In fact, the expression of the polynomial is nothing but the power series expansion around 0. This power series has only finitely many terms. An easy way to find the power series expansion about some other point is using Taylor's formula. Alternatively, we could write z as $(z - a) + a$ and substitute in the expression for $P(z)$ and expand out using the binomial theorem.

A rational function is any function of the type

$$R(z) = \frac{P(z)}{Q(z)}.$$

Unless otherwise specified, we will always assume that any common factors have been cancelled out. Set $Z(Q) := \{z \in \mathbb{C} : Q(z) = 0\}$. On $U := \mathbb{C} \setminus Z(Q)$, R is a well-defined smooth function. We will now show that these function are in fact complex-analytic on U. The points of the set $Z(Q)$ are called the *poles* of R. We will now facts about polynomials:

- **The fundamental theorem of algebra.** Any non-constant polynomial $Q \in \mathbb{C}[z]$ can be factorized to the form $c(z - a_1)^{n_1} \ldots (z - a_k)^{n_k}$ where $c \in \mathbb{C}$ and $a_i \in \mathbb{C}$ are the roots of Q.
- **The division algorithm.**If $P(z), Q(z) \in \mathbb{C}[z]$ such that $Q(z) \neq 0$, them we can find $q(z), r(z) \in \mathbb{C}[z]$ such that

$$Q(z) = q(z)P(z) + r(z), \quad \deg(r(z)) < \deg(Q(z)).$$

The above two results imply that $R(z)$ is nothing but a finite sum of polynomials and simple fractions of the type $\alpha \cdot (z - b)^{-k}$ where each b is root of Q and consequently a pole of R. It suffices to prove that each such $(z-b)^{-k}$ has a power series expansion around a point $a \in U$. Set $r = |a-b| > 0$. We have

$$\frac{1}{z-b} = \frac{1}{(z-a)-(b-a)} = \frac{1}{\left(\frac{z-a}{b-a}-1\right)(b-a)} = -\sum \frac{(z-a)^n}{(b-a)^{n+1}}.$$

The power series expansion of $\frac{1}{(z-a)^k}$ is can be obtained by differentiating the above series term-by-term k-times.

3.2 The exponential function

Let

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

where the radius of convergence of the series is ∞. To see thus note that for each $z \in \mathbb{C}$, we have:

$$\frac{|z|^{n+1}/(n+1)!}{|z|^n/n!} = \frac{|z|}{n} \to 0,$$

and we conclude by ratio test that the series $\sum_n |z|^n/n!$ is convergent for each $z \in \mathbb{C}$.

Furthermore,

$$\begin{aligned} e^z \cdot e^w &= \sum_n \frac{z^n}{n!} \cdot \sum_m \frac{w^m}{m!} \\ &= \sum_k \frac{1}{k!} \sum_{n+m=k} \frac{k!}{n!m!} z^n w^m = \sum_k \frac{(z+w)^k}{k!} = e^{z+w}. \end{aligned}$$

Hence, $e^z = e^x \cdot e^i y$, when $z = x + iy$. Now,

$$e^{iy} = \sum_n \frac{i^n y^n}{n!}.$$

The even terms give the real part of e^{iy} and the odd terms give the imaginary part. Hence,

$$\begin{aligned} \mathrm{Re}(e^{iy}) &= \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k}}{(2k)!} = \cos y \\ \mathrm{Im}(e^{iy}) &= \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k+1}}{(2k+1)!} = \sin y \end{aligned}$$

This proves that $e^{iy} = \cos y + i \sin y$ and our definition of e^z is in agreement with the earlier definition of e^{it}. This also motivates the following definitions of the trigonometric functions

$$\cos z := \frac{e^{iz} e^{-iz}}{2}, \qquad \sin z := \frac{e^{iz} - e^{-iz}}{2}$$

$$\cosh z = \frac{e^z + e^{-z}}{2}, \qquad \sinh z = \frac{e^z - e^{-z}}{2}.$$

The following properties of the exponential are easy to see:

- $e^z \neq 0 \ \forall z \in \mathbb{C}$.
- $e^{-1/z} = 1/e^z$.
- The exponential function is $2\pi i$-periodic. In particular, the equation $e^z = 1$ has solution set $2\pi i\mathbb{Z}$.
- If $e^z = e^w$ then $z - w \in 2\pi i\mathbb{Z}$.
- $|e^z| = e^{\mathsf{Re} z}, \mathsf{Im} z \in \arg e^z$.

The above properties of the exponential easily imply that the exponential function is a bijective map from the strip $B = \{z : -\pi < \mathsf{Im} z \leqslant \pi\}$ to $\mathbb{C} \setminus \{0\}$ and that it is injective on all horizontal strips of width less than 2π.

3.3 Logarithms

Definition 24. For $0 \neq z \in \mathbb{C}$, we define $\log z$ to be any complex number w such that $e^w = z$.

Just as in the case of $\arg z$, this is "multi-valued" function and as before we are interested in continuous branches. Now the exponential function restricted to $\mathbb{R}$ is a bijective mapping onto $\mathbb{R}^+$ and hence has an inverse, the natural logarithm, $\mathrm{Log} : \mathbb{R}^+ \to \mathbb{R}$. If $e^w = z$, then $e^{\mathsf{Re} w} = |z|$ and $e^{\mathsf{Im} w} \in \arg z$. Thus $\mathsf{Re} w = \mathrm{Log}\,|z|$ and $\mathsf{Im} w$ could be any of the possible values of $\arg z$. This shows that any two possible values of $\log z$ differ by an integer multiple of 2π.

Definition 25. We define the *principal branch of the logarithm* by

$$\mathrm{Log}\, z = \mathrm{Log}\,|z| + i\,\mathrm{Arg}\, z.$$

Thus, $\mathrm{Log} : \mathbb{C} \setminus \{0\} \to \mathbb{C}$. Moreover, the function $z \mapsto \mathrm{Log}\,|z|$ is a continuous function and thus the discontinuity of Log arises from the discontinuities of Arg. It should be clear that we can define the notion of the *continuous branch of the logarithm* exactly as we did before for the argument function. It is obvious that a domain U has a continuous branch of the logarithm iff it has a continuous branch of the argument function.

3.4 Complex powers

We have already seen how to define powers z^q, where q is a rational number. Now, we generalize the same definition to arbitrary w. To do this, observe that if $u = z^w$ and w is an integer then $\log z^w$ satisfies the property that $e^{w \log z} = (e^{\log z})^w = z^w + 2\pi\mathbb{Z}i$.

Definition 26. Let $z, w \in \mathbb{C}, z \neq 0$, we define z^w to be the set of complex numbers $\{e^{w \log z}$ where $\log z$ is any branch of the logarithm of z.

4 Linear fractional transformations

4.1 The Riemann Sphere

As in the case of $\mathbb{R}$, it is often convenient to add a point ∞ to the complex plane. We will perform a standard construction from topology called the Alexandroff one-point compactification. Essentially, we add a new point ∞ to $\mathbb{C}$ and call this space the *extended complex plane* or *the Riemann sphere* denoted $\widehat{\mathbb{C}}$. We equip $\widehat{\mathbb{C}}$ with a topology that makes it a compact topological space and such that $\mathbb{C} \subset \widehat{\mathbb{C}}$ is a subspace.

Definition 27. The topology of $\mathbb{C}^* := \mathbb{C} \cup \{\infty\}$ is given by the set $\mathcal{B}$ that consists of any basis of $\mathbb{C}$ and the sets of the form

$$\{z : |z| > r\} \cup \{\infty\}.$$

It is clear that $\mathbb{C}^*$ contains $\mathbb{C}$ as a subspace. The compactness of $\mathbb{C}^*$ follows easily from the fact that any neighbourhood of ∞ must necessarily contain an open set of the form $\{z : |z| > r\}$, the complement of which is certainly compact.

We will now give a concrete realization of the space $\mathbb{C}^*$. Let $S^2 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$ and let $N := (0, 0, 1)$ be the north pole. We consider $\mathbb{C}$ as the equator plane of S^2. In particular, the boundary of the unit $D \subset \mathbb{C}$ is the equator of S^2. Given a point $P = (X, Y, Z) \in S^2$, we let $z = x + iy \sim (x, y, 0)$ be the point in $\mathbb{C}$ that lies on the straight line that contains P and N. This mapping $\pi : P \mapsto z$ is called the stereographic projection. To find formula, we first observe that the parametrically the line joining N to P has the equation $N + t(P - N), t \in \mathbb{R}$. Thus the image of P under the stereographic projection must satisfy

$$(x, y, 0) = (0, 0, 1) + t((X, Y, Z - 1)),$$

which means that $1 + t(Z - 1) = 0$ or $t = \frac{1}{1-Z}$. Substituting for t, we get $x = \frac{X}{1-Z}$ and $y = \frac{Y}{1-Z}$.

On the other hand, starting with a point $z = x + iy \sim (x, y, 0) \in \mathbb{C}$, we can solve for X, Y, Z in terms of x and y. We have $X^2 + Y^2 + Z^2 = 1$ and multiplying this equation by t^2 and substituting $tX = x$ and $tY = y$ and $tZ = t - 1$, we see that $x^2 + y^2 + t^2 - 2t + 1 = t^2$ which shows $= \frac{|z|^2+1}{2}$. Thus,

$$X = \frac{2x}{1 + |z|^2}$$
$$Y = \frac{2y}{1 + |z|^2}$$
$$Z = \frac{|z|^2 - 1}{|z|^2 + 1}.$$

The above computations proves that the stereographic projection is an injective and surjective mapping from $S^2 \setminus N \to \mathbb{C}$. Moreover, S^2 is a metric space where the metric is induced from $\mathbb{R}^3$. Both the stereographic projection and its inverse are continuous and hence $\mathbb{C}$ has been embedded into S^2. The north pole is to be thought of as the point at ∞ and we can now identify $\mathbb{C}^*$ with S^2.

The following properties of stereographic projection are geometrically obvious:

1. It maps the southern hemisphere to the unit disk with the south pole getting mapped to 0.
2. The northern hemisphere minus the north pole gets mapped to the complement of the unit disk.
3. The equator gets mapped to the unit circle.

We also have the following obvious geometric fact. Our proof is analytic. For a proof using classical geometry, see Bill Casselman's article.

Theorem 28. *Under stereographic projection, circles on S^2 corresponds to circles or straight lines in the complex plane.*

Proof. The locus of points that satisfy a quadratic equation of the form

$$x^2 + y^2 + ax + by + c = 0$$

is either a circle, a point or the empty set. To see this, complete the squares,

$$(x + a/2)^2 + (y + b/2)^2 = a^2/4 + b^2/4 - c,$$

and note the three possibilities correspond respectively to $a^2/4 + b^2/4 - c$ being strictly positive, zero or strictly negative.

Any circle on S^2 is nothing but the intersection of a plane $AX + BY + C = D$ with S^2. Then the sterographic projection of this circle consists of the complex numbers $z = x + iy$ that satisfy

$$A\frac{2x}{|z|^2+1} + B\frac{2y}{|z|^2+1} + C\frac{|z|^2-1}{|z|^2+1} = D.$$

Rewriting,

$$(C - D)(x^2 + y^2) + 2Ax + 2By - (C + D) = 0$$

If $C = D$, the locus is a straight line. If $C \neq D$, then dividing by $(C - D)$, we obtain a quadratic and the locus must be a circle as it cannot be empty.

Conversely, starting with a circle on the plane

$$x^2 + y^2 + A'x + B'y + D' = 0,$$

we set $2A = A', 2B = B', C - D = 1, -(C + D) = D'$, and the corresponding set on the sphere is the intersection of the sphere with the plane $AX + BY + CZ = D$ which cannot be empty and hence must be a circle. Similarly, a straight line has the equation

$$A'x + B'y = D'$$

which determines a circle on the sphere given by the intersection of the sphere with the plane $AX + BY + CZ = D, 2A = A', 2B = B', C = D = D'/2$ and this plane meets the sphere in a circle that passes through the north pole. □

We will consider straight lines as circles through the point infinity. We end this section with some remarks on other ways we could carry out stereographic projections. Call a sphere S in $\mathbb{R}^3$ admissible if its north pole lies in the upper half-space $H = \{x_3 > 0\}$, and, for such spheres, denote by P_S the stereographic projection from the north pole N_0 of S, which identifies $\mathbb{C}^*$ with S.

4.2 Linear fractional transformations

The fractional linear transformations are the rational mappings of the form

$$Tz := \frac{az+b}{cz+d}, \ ad-bc \neq 0.$$

Note that if $ad-bc=0$, then

$$\frac{adz+bd}{d(cz+d)} = \frac{b}{d}.$$

Thus, the condition $ad-bc \neq 0$ ensures that that T is a non-constant mapping.

We can consider T as a mapping from $\mathbb{C}^*$ to itself by setting $T(-d/c)=\infty$ and $T(\infty)=a/c$. Thus T is a surjective mapping of $\mathbb{C}^*$. Furthermore, an easy computation shows that T is invertible with inverse

$$\frac{dw-b}{-cw+a}.$$

It is also easy to see that the composition of two linear fractional transformations is also a linear fractional transformation. Thus the collection of all such transformations is group called the *linear group*. Consider the following special linear fractional transformations

- *Translations:* $z \mapsto z+c$;
- *Inversion:* $z \mapsto 1/z$;
- *Homothety:* $z \mapsto \alpha z$.

It is possible to write every linear fractional transformations as a composition of the above three simple mappings:

$$\frac{az+b}{cz+d} = \frac{bc-ad}{c^2(z+d/c)} + \frac{a}{c}$$

Let S and S' be admissible spheres, and also a rigid motion T of $\mathbb{R}^3$ such that $S' := TS$ is also admissible, i.e., $TN_0 \in H$. Consider the composition $T_{S',S} = P_{S'} \circ T \circ P_S^{-1}$ which maps $\mathbb{C}^*$ to itself. In fact, every linear fractional transformation can be obtained this way.

Theorem 29. *A complex mapping is a linear fractional transformation if and only if it can be obtained by stereographic projection of the complex plane onto an admissible sphere, followed by a rigid motion of the sphere which maps it to another admissible sphere, followed by stereographic projection back to the plane.*

Proof. We write the linear fractional transformation F as

$$F(z) = \frac{\rho e^{i\theta}}{z+\alpha} + \beta.$$

If S is any admissible sphere, then translating the sphere by α gives us another admissible sphere S' and it is obvious that $T_{S',S}$ is nothing but translation by α.

To obtain rotation, dilations and inversion, we choose S to be the unit sphere. Rotations corresponds to the same rotations of $\mathbb{R}^3$ along the x_3-axis. To obtain a dilation by ρ, we move the sphere upwards by distance $\rho - 1$. And to obtain inversion, we rotate the sphere around the real-axis by an angle π. This proves the claim. $\square$