Exponential and Trigonometric functions

Our toolkit of concrete holomorphic functions is woefully small. We will now remedy this by introducing the classical exponential and trigonometric functions using power series.

1 The Exponential function

Definition 1. We define

$$\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

The fact that the series converges on the whole of $\mathbb{C}$ to a holomorphic function was shown in the previous chapter. The famous product law for the exponential now follows from our discussion on the Cauchy product.

Proposition 2. $\exp(z+w) = \exp(z)\exp(w)$.

Proof. Using the Cauchy product, we see that:

$$\exp(z)\exp(w) = \sum_k \sum_{n+m=k} \frac{z^n w^m}{n!m!} = \sum_k \frac{1}{k!}(z+w)^k = \exp(z+w).$$

In the intermediate step, we used the binomial theorem. □

The following theorem captures all the familiar properties of the exponential function

Theorem 3. *For all $z, w \in \mathbb{C}$:*

1. $\exp(z) \neq 0$;
2. $\exp(-z) = \frac{1}{\exp(z)}$;
3. $\exp|_{\mathbb{R}}$ *is a positive and strictly increasing function;*
4. $\overline{\exp(z)} = \exp(\overline{z})$;
5. *if* $z \in \mathbb{R}$ *then* $|exp(iz)| = 1$;
6. $|\exp(z)| = \exp(Rez) \leq \exp(|z|)$.

Proof. The proof consists of a number of simple checks:

1. Observe that $\exp(0) = 1$ from which it follows that $\exp(z)\exp(-z) = 1$.
2. Follows from product law.
3. If $x > 0$ then $\exp(x) > 1 + x$. If $x < 0$, then $\exp(x) = 1/\exp(-x) > 0$ from the previous part. If $y > x$ then $\exp(y) = \exp(x)\exp(y-x) > \exp(x)$.
4. Obvious.
5. $|\exp(iz)|^2 = \exp(iz) \times \overline{\exp(iz)} = \exp(0) = 1$.
6. if $z = x + iy$ then $|\exp(z)| = |\exp(x)\exp(iy)| = |\exp(x)| = \exp(x)$.

□

2 Trigonometric functions

We define the following two holomorphic functions:

Definition 4.

$$\cos(z) = \frac{\exp(iz) + \exp(-iz)}{2};$$
$$\sin(z) = \frac{\exp(iz) - \exp(-iz)}{2i}.$$

Remark 5. The definition above does not make any mention of angles. We will reverse-engineer this definition and define angles in terms of trigonometric functions.

The following properties of the trigonometric follows directly from the basic properties of the exponential function.

Theorem 6. *For all $z, w \in \mathbb{C}$:*

1. *(Euler's formula)* $\exp(iz) = \cos(z) + i\sin(z)$*;*
2. $\cos(z)$ *is an even function and* $\sin(z)$ *is an odd function;*
3. $\cos^2(z) + \sin^2(z) = 1$*;*
4. $\sin(z + w) = \sin(z)\cos(w) + \cos(z)\sin(w)$*;*
5. $\cos(z + w) = \cos(z)\cos(w) - \sin(z)\sin(w)$*;*
6. $\sin(z) = \sum \frac{(-1)^{2n+1} z^{2n+1}}{(2n+1)!}$*;*
7. $\cos(z) = \sum \frac{(-1)^n z^{2n}}{(2n)!}$
8. $\sin'(z) = \cos(z), \cos'(z) = \sin(z)$*.*

Caution: The above theorem might give the impression that the trigonometric functions defined above behave exactly the same as the classical trigonometric functions. However, this is untrue for complex numbers. For instance, if $y > 0$ then it is easy to see that

$$\cos(iy) = \frac{\exp(y) + \exp(y)}{2} > \frac{(1 + y)}{2},$$

which shows that cos and hence sin is unbounded on $\mathbb{C}$. In fact, we will show later in the course that any bounded entire function is forced to be constant!

3 Periodicity and the definition of π

As we have deliberately avoided all mention of angles so as to have a self-contained development, the properties of the number π are unavailable to us. In fact, we are yet to understand the behavior of the trigonometric functions on the real-axis. We will now show that sin and cos are periodic function with period 2π (we will define π shortly). We will also show that the circumference of the unit circle is 2π.

Let us assume for the moment that we have found $t \in \mathbb{C}$ such that $\sin(t) = 0$. Then from the trigonometric identities, we see that

$$\cos(2t) = 1 - 2\sin^2(t) = 1, \ \sin(2t) = 0,$$

which means that $\sin(z + 2t) = \sin(z)$. Similarly, we can show that $2t$ is a period of cos as well. Thus, we are turn our attention to determining the zeroes of sin.

Let S denote the set of zeroes of sin. If $z \in S$ then

$$\exp(iz) = \exp(-iz) = 0$$

and so $\exp(2iz) = 1$ from which it follows that $\exp(-2y) = |\exp(2iz)| = 1$. This forces y to be real. We have shown that $S \subset \mathbb{R}$. Now, we need an useful inequality.

Lemma 7. *If* $0 < x < 2$ *then we have*

$$\sin(x) > x(4 - x^2)/4 > 0.$$

Proof. We have from the power series expansion that

$$\begin{aligned}
|\sin(x) - x| &\leq \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \\
&\leq \frac{x^3}{3!}\left(1 + \left[\frac{x^2}{20}\right] + \left[\frac{x^2}{20}\right]^2 + \ldots\right) \\
&\leq \frac{x^3}{3!}\left(1 + \frac{1}{5} + \frac{1}{5^2} + \ldots\right) \\
&< x^3/4,
\end{aligned}$$

Hence

$$x - \sin(x) < x^3/4$$

from which the result follows. □

From the above inequality, it follows that $\sin(1) > 3/4$ and therefore $\cos(1) < 3/4$. We apply the intermediate value theorem on the interval $[0, 1]$ to the function $\sin - \cos$ to conclude that $\sin(t) = \cos(t)$ for some $t \in [0, 1]$. Using the trigonometric identities $\cos(2t) = \sin(4t) = 0$. This shows that sin has a zero in $[0, 4]$. Clearly, S is a subgroup. Any additive subgroup of $\mathbb{R}$ is either dense or cyclic. Clearly, S is not dense. Let π be the generator of S. From the inequality above, $\pi > 2$ and the discussion above shows $\pi < 4$. This means that $0 < \pi/2 < 2$ and therefore $\sin(\pi/2) > 0$. But

$$0 = \sin(\pi) = 2\sin(\pi/2)\cos(\pi/2)$$

which along with the identity $\sin^2(\pi/2) + \cos^2(\pi/2) = 1$ forces $\sin(\pi/2) = 1$ and $\cos(\pi/2) = 0$. This implies that

$$\sin(\pi/2 + x) = \sin(\pi/2 - x)$$

and as sin is positive on $[0, \pi/2]$, it follows that sin is positive on $(0, \pi)$. It now follows from the identity

$$\sin(y + x)\sin(y - x) = \sin^2(y) - \sin^2(x)$$

that if $0 < x < y < \pi/2$ then $\sin^2(y) > \sin^2(x)$ and hence sin is an increasing function on $[0, \pi/2]$. If p is a period of $\sin(z)$ then

$$\sin(z) = \sin(z + p) = \sin(z)\cos(p) + \cos(z)\sin(p).$$

If $z = 0$ then $p = n\pi, n \in \mathbb{Z}$. However as sin is an odd function, the period of sin must be $2n\pi$. In a similar way, we can prove that the periods of cos is also $2n\pi$. We can summarize this discussion succinctly in the following:

Theorem 8. *The following are equivalent:*

1. p *is a period of* exp*;*
2. $\exp(p) = 1$*;*
3. $p \in \{2n\pi i : n \in \mathbb{Z}\}$.

Henceforth, we shall freely use the basic properties of trigonometric functions without fuss.

4 Argument of a complex number

Given any non-zero complex number z, we can define the unit vector $z/|z| = u + iv$. Now $|u| \leq 1$, we can find a unique $\theta \in [0, \pi]$ such that $\cos(\theta) = 1$. This is because cos is a strictly-decreasing function from 1 to -1 in this interval. Now, either $\sin(\theta) = v$ or $\sin(\theta) = -v$. If $\sin(\theta) = -v$, we replace θ by $-\theta$. In either case, we have found a θ such that $z/|z| = \cos(\theta) + i\sin(\theta)$. Thus, using Euler's theorem, we can write

$$z = |z| \exp(i\theta),$$

which is called the **polar representation** of the number z. The "angle" θ is called the *argument* of the number z. This is the angle the vector z makes with the x-axis. The choice of θ—as one would anticipate—is unique only up to an addition by $2n\pi$ by the periodicity of sin and cos.

Definition 9. Given $z \neq 0$, we define

$$\operatorname{Arg}(z) := \{\theta \in \mathbb{R} : z = |z| \exp(i\theta)\}.$$

Note that the capital letter in Arg is intentional and indicates that the object is a set and not a number. When we use the notation $\arg(z)$, we mean an arbitrary fixed choice made from $\operatorname{Arg}(z)$.

By considering a unit vector z, we can form the triangle using the x-axis and the vector z and a line parallel to the y-axis. It is now clear that the new definitions of sin and cos are the same as the classical ones defined as the ratios of sides of right-triangles.

The argument satisfies the following properties:

Proposition 10. *Let* $z, w \in \mathbb{C} \setminus \{0\}$*. Then*

1. $\operatorname{Arg}(zw) = \operatorname{Arg}(z) + \operatorname{Arg}(w)$*;*
2. $\operatorname{Arg}(z^{-1}) = -\operatorname{Arg}(z)$*;*
3. $\operatorname{Arg}(z/w) = \operatorname{Arg}(z) - \operatorname{Arg}(w)$.

Remark 11. The algebraic signs appearing in the above proposition have to be interpreted as element-wise operations. In particular, the '$-$' sign has nothing to do with set-theoretic complements.

We can now define the general notion of an angle. Given a two complex numbers z, α with $\alpha \neq 0$, we define the ray

$$L(z, \alpha) := \{z + t\alpha : t \geq 0\}.$$

It is easy to see that if $L(z_1, \alpha_1) = L(z_2, \alpha_2)$ iff $z_1 = z_2$ and $\operatorname{Arg}(\alpha_1) = \operatorname{Arg}(\alpha_2)$. We define the angle

$$(L(z, \alpha_1), L(z, \alpha_2)) = \operatorname{Arg}(\alpha_2/\alpha_1).$$

Remark 12. Note that the notion of angle we have defined is an oriented angle. Specifically, $(L(z, \alpha_1), L(z, \alpha_2)) = -(L(z, \alpha_2), L(z, \alpha_1))$.

5 The complex logarithm

Definition 13. For $z \in \mathbb{C} \setminus \{0\}$, we define

$$\mathrm{Log}(z) := \{w \in \mathbb{C} : \exp(w) = z\}.$$

Remark 14. The logarithm is not defined for $z = 0$ for obvious reasons: the above set will be empty. Also keep in mind that, as per our notational convention, the complex logarithm is not a *function* but a **set**.

Proposition 15. *Let $z = r\exp(i\theta)$. Then*

$$\mathrm{Log}\, z := \{x + iy : \log x = r, y \in \mathrm{Arg}(z)\}.$$

Proof. If $w = x + iy$ then $\exp(w) = \exp(x)\exp(iy)$ from which the proposition follows immediately. □

6 Roots of unity

Let us determine the n-th roots of unity for $0 < n \in \mathbb{N}$. If $w^n = 1$, then $w = \exp(i\theta)$ where

$$\theta \in \left\{\frac{2k\pi}{n} : k \in \mathbb{Z}\right\}.$$

The above set is an infinite set. However, the set of *distinct* values of w that arise from the set correspond to $k = 0, 1, \ldots, n-1$. The quantity $\omega := \exp(2\pi/n)$ generates all such values: the n-th roots of unity are precisely $1, \omega, \omega^2, \ldots, \omega^{n-1}$. All the basic properties about the n-th roots can be easily proved. The n-th roots of an arbitrary complex number can also be easily computed.

We have already defined π using the sin function. We relate this to the circumference of the circle. Observe that the n-th roots of unity form a regular polygon. The circumference of this polygon is clearly

$$L_n = n|1 - \omega| = n|1 - (\cos(2\pi/n) + i\sin(2\pi/n))|.$$

A strightforward computation now gives

$$|1 - \omega|^2 = 2(1 - \cos(2\pi/n)) = 4\sin^2(\pi/n).$$

This shows that $L_n = 2n\sin(\pi/n)$. Recall that if $0 < x < 2$, we have

$$|\sin(x) - x| < x^3/4.$$

Setting $x = \pi/n$, we see that

$$|L_n - 2\pi| < \pi^3/2n^3.$$

This shows that as $n \to \infty$, $L_n \to 2\pi$.