Solutions

Eneström – Kakeya theorem

Let $\quad P(z) = c_0 + c_1 z + \cdots + c_n z^n$

with $\quad c_0 > c_1 > c_2 \cdots > c_n$ and $n > 1$.

Show that if α is zero of p, then $|\alpha| > 1$.

<u>Proof:-</u> Suppose $|\alpha| \leq 1$ and $p(\alpha) = 0$

Then $\quad (1-\alpha)\, p(\alpha) = 0$.

$$(c_0 + c_1\alpha + \cdots + c_n\alpha^n)$$

$$- \; c_0\alpha - c_1\alpha^2 - \cdots - c_n\alpha^{n+1}$$

Therefore

$$c_n\alpha^{n+1} = c_0 + (c_1 - c_0)\alpha + \cdots + (c_n - c_{n-1})\alpha^n$$

$$\Rightarrow \quad c_n|\alpha|^{n+1} \geq c_0 - (c_0 - c_1)|\alpha| \; \cancel{+} \cdots - (c_{n-1} - c_n)|\alpha|^n$$

$$\geq c_0|\alpha| - (c_0 - c_1)|\alpha| - \cdots - (c_{n-1} - c_n)|\alpha|$$

(as $|\alpha| \geq |\alpha|^k$ and $c_{i-1} > c_i$)

$$= |\alpha|\Big(c_0 - (c_0 - c_1) - \cdots - (c_{n-1} - c_n)\Big)$$

$$= c_n|\alpha|$$

so

$$c_n|\alpha|^{n+1} \geq c_n|\alpha|$$

$$\Rightarrow |\alpha| = 1.$$

Now consider the new polynomial

$$Q(z) = P(rz) = c_0 + c_1 r z + c_2 r^2 z^2 + \cdots + c_n r^n z^n$$

if $r > 1$ and r is sufficiently close to 1, then $Q(z)$ also satisfies the hypothesis $c_0 > c_1 r > c_1 r^2 > \cdots > c_n r^n$

and hence any root β of Q must

satisfy $|\beta| \geq 1$. However, $\frac{|\alpha|}{r}$ is a root and therefore

$$\frac{|\alpha|}{r} \geq 1 \implies |\alpha| \geq r.$$

Suppose $f : D \to \mathbb{C}$ is $\mathbb{R}$-diff at a and $\lim_{h \to 0} \frac{|f(a+h) - f(a)|}{|h|}$ exists

Show that either f or $\bar{f}$ is $\mathbb{C}$-diff. at a.

The proof involves reformulating the defn. of $\mathbb{R}$-differentiability. Let T be the real derivative of f at a.

Then $$\lim_{h \to 0} \frac{|f(a+h) - f(a) - Th|}{|h|} = 0$$

we may remove the modulus. Note this work only for $\mathbb{R}^2 = \mathbb{C}$ as division makes sense here.

We have

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - Th}{h} = 0$$

let $f_1 = \begin{cases} \dfrac{f(a+h) - f(a) - Th}{h} & h \neq 0 \\ 0 & h = 0 \end{cases}$

Then f_1 is defined for small h and f_1 is continuous. We have

$$\begin{aligned} f(a+h) &= f(a) + Th + f_1(h)h \\ &= f(a) + \alpha h + \beta \bar{h} + f_1(h)h \end{aligned}$$

(T is $\mathbb{R}$-linear)

Now, $$\frac{|f(a+h)-f(a)|}{|h|}$$

$$= \frac{|\alpha h + \beta \bar{h} + f_1(h)h|}{|h|}$$

$$= \frac{|h(\alpha + f_1(h)) + \beta \bar{h}|}{|h|}$$

$$= \left|(\alpha + f_1(h)) + \beta \frac{\bar{h}}{h}\right|$$

We now use polar coordinates (to be introduced soon)

and set $h = re^{i\theta}$ and take $r \to 0$, to conclude that

$$\lim_{h \to 0} \frac{|f(a+h)-f(a)|}{|h|} = |\alpha + \beta e^{-2i\theta}|$$

Complete the proof!.