Holomorphic Functions and Power Series

Having studied the basic properties of the complex field, we now study functions of a complex variable. We define what it means for a complex-valued function defined on a domain in the complex plane to be complex-differentiable. We then study the relationship between this new notion of differentiability and the already familiar notion of real-differentiability. Functions that are complex-differentiable at every point in a domain are called holomorphic functions. For various reasons, which we shall study in great detail, a function being holomorphic is far more restrictive than the function being real-differentiable.

The focal point of this chapter will be power series and functions that have local power series expansions, i.e., analytic functions. The reason for this is two-fold:

1. The most important functions that arise in practice are analytic.
2. All holomorphic functions are analytic!

The second point above will be one of the central theorems we will prove in this course. We end this chapter with a brief exposition of the exponential and trigonometric functions.

Notation: Throughout this chapter, U will be a domain in $\mathbb{C}$ with $a \in U$ and $f : U \to \mathbb{C}$. will be a function.

1 Holomorphic functions

1.1 Definitions and examples

Definition 1. Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be a map and $a \in U$. We say f is *complex-differentiable (ℂ-differentiable at a)* if

$$\lim_{0 \neq z \to a} \frac{f(z) - f(a)}{z - a}$$

exists and in which case we denote the limit by $f'(a)$ and call it the *complex derivative* of f at a.

We say that f is *holomorphic* on U if f is $\mathbb{C}$-differentiable at every point of U. A holomorphic function on $\mathbb{C}$ is called an *entire* function.

Remarks.

1. It is easy to see that if f is $\mathbb{C}$-differentiable at a then f is continuous at a.
2. If f and g are $\mathbb{C}$- differentiable then so are $f \pm g, fg$ and $cf, c \in \mathbb{C}$. The proofs of these facts follows *mutatis mutandis* from the proof in the real case.
3. If f is $\mathbb{C}$-differentiable at a and $f'(a) \neq 0$, then $\frac{1}{f}$ is $\mathbb{C}$- differentiable at a and its derivative at a is $\frac{-1}{f'(a)^2}$.

Examples and non-examples

We will provide several examples and non-examples to facilitate the digestion of the definitions of $\mathbb{C}$-differentiability and holomorphicity.

1. The function z^n is entire with derivative nz^{n-1}. Combined with fact that sums and products of $\mathbb{C}$-differentiable functions are $\mathbb{C}$-differentiable we see that any polynomial in z is entire.
2. Quotients of $\mathbb{C}$-differentiable functions are $\mathbb{C}$-differentiable wherever they are defined.
3. The function $f(z) = \overline{z}$ is not $\mathbb{C}$-differentiable at any point. To see this, observe that the definition gives
$$\frac{f(z_0 + h - f(z_0))}{h} = \frac{\overline{h}}{h}$$
which has different limits as $h \to 0$ along the real and imaginary axes.
4. Consider the function $f(z) = |z|^2 = z \cdot \overline{z}$. In real coordinates, we can consider at as the function $(x, y) \mapsto x^2 + y^2$. This is polynomial function in x, y and hence C^∞-smooth. Again computing as before, we see that the difference quotient if of the form $\overline{a} + a\overline{h}/h + \overline{h}$. As $h \mapsto 0$, the above difference quotient has a limit iff $a = 0$. Hence $|z|^2$ is $\mathbb{C}$-differentiable only at the point 0.

The last two examples show that f being $\mathbb{C}$-differentiable at a point a is a severe restriction. The limit of the difference quotient must exist and agree on all possible approaches to the point a. We could approach a along any line, or even complicated curves like spirals and the limiting value must exist and agree on any such approach. This is the crucial difference between real-differentiability of a function defined on an interval and $\mathbb{C}$-differentiability. We will now explore this connection in some detail.

1.2 Relationship between the complex derivative and the real derivative

Let f be $\mathbb{C}$-differentiable at a. Let $R(h)$ (remainder) be the function defined as follows:

$$R(h) = \begin{cases} \frac{f(a+h)-f(a)}{h} - f'(a) & \text{if } h \neq 0 \\ 0 & \text{if } h = 0 \end{cases}.$$

Observe that R is well-defined for h sufficiently close to 0 and that R is continuous. The definition of $\mathbb{C}$-differentiability can now be restated as follows: f is $\mathbb{C}$-differentiable at a iff we can find a complex number α and a continuous function $R : D \to \mathbb{C}$, where $D \subset \mathbb{C}$ is a suitably small neighborhood of 0 such that $R(0) = 0$ and

$$f(a + h) = f(a) + \alpha h + R(h)h.$$

Recall that f is $\mathbb{R}$-differentiable at a if we can find a linear transformation $df_a : \mathbb{R}^2 \to \mathbb{R}^2$ such that:

$$\lim_{h \to 0} \frac{\|f(a + h) - f(a) - df_a h\|}{\|h\|} \to 0.$$

If f is $\mathbb{C}$-differentiable at a then clearly f is $\mathbb{R}$-differentiable at a with the role of df_a played the $\mathbb{C}$-linear (and hence $\mathbb{R}$-linear map) $h \mapsto f'(a)h$. Furthermore, if f is $\mathbb{R}$-differentiable at a and the derivative df_a is $\mathbb{C}$-linear then from our discussion in the previous chapter, it follows that df_a is of the form $h \mapsto \alpha h$ for some complex number $\alpha \in \mathbb{C}$. Hence, f is $\mathbb{C}$-differentiable at a.

1.3 The Cauchy–Riemann equations

Let U, f, a be as before and write z as $x + iy, x, y \in \mathbb{R}$ and $f = u + iv$ where u and v are real-valued. Assume that a, the partial derivatives u_x, u_y, v_x, v_y exist. We define the Wirtinger derivatives

as follows:

$$\frac{\partial f}{\partial z}(a) := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)(a)$$

$$\frac{\partial f}{\partial \bar{z}}(a) := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)(a).$$

Remark 2. Contrast the definition above with the second characterization of an $\mathbb{R}$-linear map given in the last chapter.

Theorem 3. *With notations as above, the following conditions are all equivalent.*

1. f is $\mathbb{C}$-differentiable at a.
2. f is real-differentiable at a and $\frac{\partial f}{\partial y}(a) = i\frac{\partial f}{\partial x}(a)$.
3. f is real-differentiable at a and $\frac{\partial f}{\partial \bar{z}}(a) = 0$.
4. f is real-differentiable at a and $u_x = v_y$ and $u_y = -v_x$.

Proof. Let us see how the map df_a acts on the two elements $1, i \in \mathbb{C}$. Now $1, i$ are nothing but the vectors $(1, 0)$ and $(0, 1)$, respectively, in $\mathbb{R}^2$. This $df_a(1) = \frac{\partial f}{\partial x}(a)$ and $df_a(i) = \frac{\partial f}{\partial y}(a)$. Now, df_a is $\mathbb{C}$-linear iff $df_a(i) = idf_a(1)$ and this proves the equivalence. □

Remark 4. The above proof is very slick but comes with the risk of deluding you into a false sense of understanding. Please mull over the proof keeping in mind the discussion on linear mappings in the last chapter.

2 Power series

Our development of the theory is so far severely deficient in examples. We remedy this situation with a dose of power series. We first recall some deeper facts about convergent infinite series.

2.1 Infinite series

We will assume that the reader is sufficiently adept at identifying whether a (complex) series is (absolutely) convergent or not using the standard convergence tests such as the comparison test, root test, ratio test, etc.

We will now sketch the proof of one-part of a famous theorem of Riemann on the behavior of absolutely convergent series.

Theorem 5. *Let $\sum z_n$ be an absolutely convergent series. Then given any bijection $\sigma : \mathbb{N} \to \mathbb{N}$, the new series $\sum z_{\sigma(n)}$ is convergent and has the same sum as $\sum z_n$.*

Proof. Let s_n denote the n-th partial sum of the series $\sum z_n$ and s'_n that of the series $\sum z_{\sigma(n)}$. Let $\varepsilon > 0$. First choose N_0 so large so that $\sum_{n=N_0}^{\in} |z_n| < \varepsilon$ and then choose $N_1 > N_0$ so that $\sigma(\{1, 2, \ldots, N_1\}) \supset \{1, 2, \ldots, N_0\}$. Then it is clear that $|s_n - s'_n| < \varepsilon$ proving that the series $\sum z_{\sigma(n)}$ converges to the same sum as $\sum z_n$. □

Definition 6 (Cauchy product). Let $\sum a_n$ and $\sum b_n$ be two series. We define the Cauchy product of the two series to be the new series $\sum c_k$, where $c_k = \sum_{n+m=k} a_m b_n$.

Theorem 7. *If the series $\sum a_n$ and $\sum b_n$ are both absolutely convergent then so is their Cauchy product. Moreover the Cauchy product converges to the product of the sums of the two series.*

Proof. Let $\sum |a_n| = \alpha$ and $\sum |b_n| = \beta$. Then observe that

$$\sum_{k=0}^{N} |c_k| \leq (|a_0| + |a_1| + \cdots + |a_N|)(|b_0| + |b_1| + \ldots |b_N|) \leq \alpha\beta.$$

This shows that the Cauchy product is absolutely convergent. Now,

$$\left| \sum_{k} c_k - \sum_{n=0}^{N} (a_0 + a_1 + \cdots + a_n)(b_0 + b_1 + \cdots + b_n) \right| \leq \alpha \sum_{k=N+1}^{\infty} |b_k| + \beta \sum_{K=N+1}^{\infty} |a_k|.$$

The last term above can be made as small as desired. $\qquad\square$

2.2 Power series and radius of convergence

Definition 8. A *formal power series about $z_0 \in \mathbb{C}$ with coefficients $c_n \in \mathbb{C}$* is a series of the form

$$\sum_{n \in \mathbb{N}} c_n (z - z_0)^n.$$

Remark 9. We may substitute different complex numbers in the place of z to convert the above formal series to an honest one. However, apart from $z = z_0$, it is not at all obvious that such a substitution leads to a convergent series.

Given a formal power series $\sum_{n \in \mathbb{N}} c_n (z - z_0)^n$, let us consider the set of convergence

$$\mathscr{C} := \{ z \in \mathbb{C} : \sum_{n \in \mathbb{N}} c_n (z - z_0)^n \text{ converges} \}.$$

Our objective now is to study the geometry of the set $\mathscr{C}$. Define

$$R := \sup\{R \in \mathbb{R}_+ : \exists M_R \; \forall n \in \mathbb{N} \; |a_n| R^n < M_R\}.$$

Theorem 10. *Given a formal power series $\sum_{n \in \mathbb{N}} c_n (z - z_0)^n$ and R as defined as above, the series converges absolutely for $z \in D(z_0, R)$ and diverges whenever $|z| > R$. Moreover, if $f : D(z_0, R)$ is defined by the sum of the series then sequence $f_k := \sum_{n=0}^{k} c_n (z - z_0)^n$ converges uniformly on compact subsets of $D(z_0, R)$ to f.*

Proof. If $|z| > R$ then the series cannot converge as the n-th term does not go to 0. On the other hand, if $|z| < r < R$ we have

$$\sum |c_n| |z - z_0|^n = \sum r^n |c_n| \left(\frac{|z - z_0|}{r} \right)^n \leq M_r \sum \left(\frac{|z - z_0|}{r} \right)^n.$$

By comparison test, the above series converges. By Weierstrass M-test, the assertion regarding uniform convergence on compacts follows.

$\qquad\square$

Examples.

1. The geometric series $1 + z + z^2 + \ldots$ converges to the function $\frac{1}{1-z}$ whenever $|z| < 1$. The behaviour on the unit circle is very simple in this case. We have divergence at all points since the n-th term does not go to 0.

As one expects, power series can be differentiated term-by-term.

Theorem 11. *Let $\sum c_n(z - z_0)^n$ be a convergent power series with radius of convergence $R > 0$ and let $f : D(z_0, R) \to \mathbb{C}$ be the sum. Then f is holomorphic on $D(a, R)$ and its derivative is given by*

$$\sum n c_n (z - z_0)^{n-1}.$$

Proof. First of all, the derived power series has radius of convergence R. To see this, observe that if $n|c_n|r^{n-1} < M_r$ then $|c_n|r^n < rM_r$. showing that $R' \leq R$ (R' is the radius of convergence of the derived power series). On the other hand, if $0 < r < \rho < R$ then

$$n|c_n|r^{n-1} = n/r|c_n|\rho^n \left(\frac{r}{\rho}\right)^n \leq n/rM_\rho(\left(\frac{r}{\rho}\right)^n)$$

But, $n\left(\frac{r}{\rho}\right)^n$ is a bounded sequence and therefore we can find an upper bound for $n|c_n|r^{n-1}$ showing that $R \leq R'$.

Now, we must show that for each $\varepsilon > 0$ and $z \in D(z_0, R)$, we have

$$\left| \frac{f(z) - f(w)}{z - w} - \sum_n c_n(z - z_0)^{n-1} \right| < \varepsilon$$

when w is in a suitably small neighborhood of z. To see this, we employ a standard technique in analysis: break up the series into the first few terms and the tail which we know can be made very small. To this end, write $f(z) = S_N(z) + R(z)$ where S_N the partial sum of the power series of f. We will choose N suitably to force the above required inequality. The difference quotient becomes

$$\left(\frac{S_N(z) - S_N(w)}{z - w} - \sum_{n=0}^{N} n c_n(z - z_0)^{n-1} \right) - \sum_{n>N} n c_n(z - z_0)^{n-1} + \left(\frac{R_N(z) - R_N(w)}{z - w} \right).$$

Call the three terms I, II and III, respectively. The second term can be made as small as desired (being the tail of a convergent series) by choosing N to be suitably large. On the other hand, the first term can be made as small as desired by choosing w in a suitably small neighborhood of z. The only term to worry about is III. We have

$$\sum_{n>N} c_n \left((z - z_0)^{n-1} + (z - z_0)^{n-2}(w - z_0) + \cdots + (w - w_0)^{n-1} \right)$$

where we expanded out $(z - z_0)^{n-1} - (w - z_0)^{n-1}$ and canceled the $z - w$ in the denominator. We may choose z and w so that $|z - z_0| < r < R$ and $|w - z_0| < R$ whence the above sum becomes

$$\leq \sum_{n>N} n c_n \rho^{n-1}$$

which is the tail of a convergent series and can be made as small as desired by choosing N suitably large (here it is irrelevant what z and w are as long as both are in $D(z_0, R)$). In conclusion, we first choose N so large that the second and third term become lesser than $\varepsilon/3$ and then we choose a small enough neighborhood of z so that I also becomes smaller than $\varepsilon/3$. $\square$

Corollary 12. *If f is the limit of a convergent power series $\sum c_n(z - z_0)^r$ in $D(z_0, R)$ then f is infinitely differentiable and we have*

$$c_n = \frac{f^n(z_0)}{n!}.$$

Proof. Straightforward. □

Many of the nice properties of holomorphic functions stem from the following deep theorem which is the central theorem of the course. We first need a definition.

Definition 13. Let $f : U \to \mathbb{C}$ be a function. We say that f is complex-analytic if for each $z_0 \in U$, we can find a power series

$$\sum c_n(z - z_0)^n$$

such that on some disk $D(z_0, r) \subset U, r > 0$, the series converges to $f|_{D(z_0, r)}$.

Theorem 14. *Any holomorphic function is automatically complex-analytic!*

We will spend the better part of two chapters developing the theory of contour integration to prove this result. After that, the rest of the course will be spent on obtaining an endless stream of wonderful results! To whet you appetite, here is one particularly famous one:

Theorem 15 (Open mapping theorem)**.** *Let $f : U \to \mathbb{C}$ be holomorphic. Then either f is constant or f is an open map.*