The Complex Numbers

In this chapter, we will study the basic properties of the field of complex numbers. We will begin with a brief historic sketch of how the study of complex numbers came to be and then proceed to develop tools needed to study calculus on the complex plane. We will also give several applications of complex numbers to solving classical problems from geometry and trigonometry.

1 Arithmetic of the complex plane

1.1 Why complex numbers?

If you recall from your study of Real Analysis, the introduction of the real numbers as the completion of the field of rational numbers is unavoidable if one wants to develop calculus in an adequate manner. The question arises as to why we need to enlarge $\mathbb{R}$ further and introduce the field $\mathbb{C}$. Complex numbers were first introduced in 1545 by the Italian mathematician Cardano in his *Ars magna* in connection with quadratic equations and he immediately discards them commenting they were "as subtle as they are useless". In fact, complex numbers were often dismissed as "imaginary" or "impossible" even by prominent mathematicians such as Leibniz. So why study them at all?

Recall from high school mathematics that the solution of the quadratic equation $x^2 = mx + c$ is given by the expression

$$x = \frac{1}{2}\left(m \pm \sqrt{m^2 + 4c}\right). \tag{1.1}$$

The quantity under the square root is called the *discriminant*, denoted D. According to most textbooks, complex numbers were introduced to ensure that quadratic equations always have solutions. This is not only historically inaccurate but also highly misleading.

Geometrically, solving the quadratic equation $x^2 = mx + c$ is same as finding the points of intersection of the parabola P given by the equation $y = x^2$ and the line L given by the equation $y = mx + c$. Three possibilities can arise,

(i) L and P intersects at two points. This corresponds to (1.1) yielding two real solutions. In this case $D > 0$.
(ii) L and P intersects at one point. In this case $D = 0$.
(iii) L and P do not intersect at all. In this case $D < 0$.

Thus, when $D < 0$, the fact that P and L do not intersect is reflected in the fact that (1.1) are complex numbers. This shows that there is absolutely no reason to introduce complex numbers in the study of quadratic equations. Cardano was perfectly justified in dismissing complex numbers as "useless" in connection to solving quadratic equations. That complex numbers were introduced to solve quadratic equations is a lie repeated blindly by many ill-informed textbook authors!

The real reason for introducing complex numbers is in connection with solving cubics. We begin with the first theorem of the course.

Theorem 1 (Cardano)**.** *The solution of the cubic equation*

$$x^3 = 3px + 2q \tag{1.2}$$

is given by

$$x = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}}. \tag{1.3}$$

Any arbitrary cubic $x^3 + ax^2 + bx + c$ can be transformed by a linear change of coordinates to an equation of the form (1.2).

Proof. It is an exercise for you to prove that an arbitrary cubic can indeed be transformed to an equation of the form (1.2).

To solve (1.2), we first set $x = u + v$. Expanding the LHS of (1.2), we get

$$u^3 + v^3 + 3uv(u + v) = u^3 + v^3 + 3uvx.$$

Equating the above equation with the RHS of (1.2), we see that $p = uv$ and that $u^3 + v^3 = 2q$. Eliminating v, we end up with the equation

$$u^3 + \frac{p^3}{u^3} - 2q = 0.$$

This is a quadratic in u^3 whose solutions are given by

$$u^3 = q \pm \sqrt{q^2 - p^3}.$$

By symmetry, v^3 has the exact same solutions. As $u^3 + v^3 = 2q$, without loss of generality, we can take as the solutions for u^3 and v^3 as

$$u^3 = q + \sqrt{q^2 - p^3}$$
$$v^3 = q - \sqrt{q^2 - p^3}$$

Thus the required solutions are given by

$$x = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}}.$$

□

Geometrically, solving for the cubic (1.2) is equivalent finding the intersection of the cubic with the line L given by the equation $y = 3px + 2q$. Note that a cubic equation *always* has at least one real root (why?). This means that the formula (1.3) must always yield at least one real number. It was Bombelli who realized that there is something strange about the formula. He considered the cubic $x^3 = 15x + 4$ which has solutions

$$x = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i},$$

here we are freely using complex notation which I am assuming you are already familiar with. The above does not seem to be a real number at all. But, Bombelli had a "wild thought". By guessing, he realized that $x = 4$ solves the cubic. So he assumed that $\sqrt[3]{2 + 11i}$ is an expression of the type $2 + ui$ and $\sqrt[3]{2 - 11i}$ is an expression of the form $2 - ui$, so that $x = 2 + ui + 2 - ui = 4$. Of course, for this to make sense Bombelli assumed that the ordinary laws of algebra are true for the complex numbers, i.e.,

$$a + ib + a' + ib' = (a + a') + i(b + b').$$

Next to determine u, he needed to evaluate $(2+ui)^3$. To do this, he assumed that multiplication obeys the following obvious rule

$$(a + ib)(a' + ib') = (aa' - bb') + i(a'b + ab'),$$

where we are using $i^2 = -1$. Expanding out $(2 + ui)^3$ using the above rule we get

$$-iu^3 - 6u^2 + 12iu + 8 = 2 + 11i,$$

which readily yields $u = 1$. Similarly, $(2 - i)^3 = 2 - 11i$.

Bombelli's "wild thought" shows that the "useless" complex numbers are unavoidable in the solution of cubics. However, the study of complex numbers remained a mere curiosity and were considered mysterious for almost 250 years.

1.2 Notation and terminology

We will identify the set of complex numbers, denoted $\mathbb{C}$, with $\mathbb{R}^2$. In this identification, the complex number $a + ib$ corresponds to the pair (a, b). The point $1 \in \mathbb{C}$ corresponds to $(1, 0)$ and the point i corresponds to $(0, 1)$. Geometrically, a complex number is nothing but a vector in the so called *Argand–Gauss* complex plane.

The following picture and table summarizes all the relevant notation and terminology related to complex numbers.

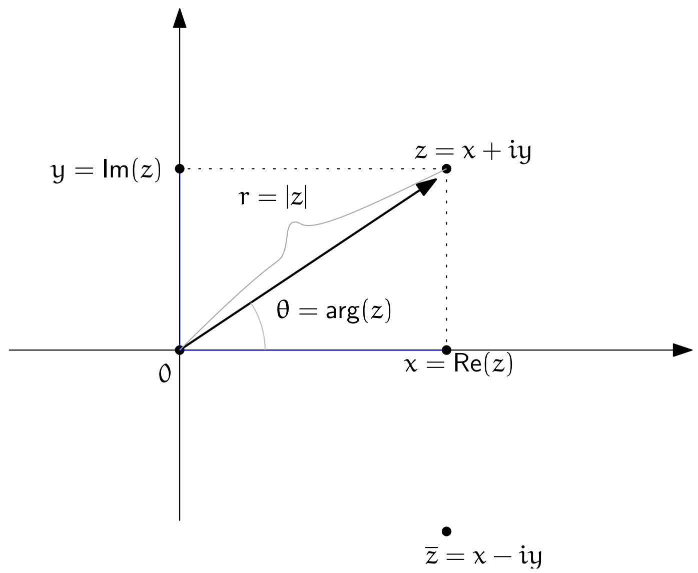

Figure 1: The complex plane

Terminology	**Meaning**	**Notation**
modulus of z	length r of the vector z	$\lvert z\rvert$
argument of z	angle θ that the vector z makes with the x-axis	$\arg(z)$
real part of z	x coordinate of the vector z	$\mathrm{Re}(z)$
imaginary part of z	y coordinate of the vector z	$\mathrm{Im}(z)$
real axis	the set of real numbers	
imaginary number	a number that is a real multiple of i	
imaginary axis	the set of imaginary numbers	
complex conjugate of z	reflection of z in the real axis	$\overline{z}$

Note that for $z = x + iy, \overline{z} = x - iy$ and

$$\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z}), \quad \mathrm{Im}(z) = \frac{1}{2i}(z - \overline{z}).$$

The sum of two complex numbers z and w can be obtained geometrically using the parallelogram law for addition of vectors.

In order to visualize the product, we need to introduce the polar representation of complex numbers in terms of r and θ. The modulus of $|z|$ is the distance from the origin to the point (x, y). Explicitly, $|z| = \sqrt{x^2 + y^2} = \sqrt{z \cdot \overline{z}}$. The modulus satisfies a number of simple and easy to prove inequalities.

$$\begin{aligned} \left|\sum_{i=1}^{n} z_i\right| &\leq \sum_{i=1}^{n} |z_i|, \\ ||z| - |w|| &\leq |z \pm w|, \\ |z_1 \cdots z_n| &\leq |z_1| \cdot |z_2| \cdots |z_n|. \end{aligned}$$

The modulus is also related

With the sum and product defined as in the previous section, $\mathbb{C}$ is a commutative field. The inverse of the number $z \neq 0$ is given by $z^{-1} = \frac{1}{z} = \frac{\overline{z}}{|z|^2}$.

We now want rigorously define the the argument of z. To do this, we have to introduce trigonometric functions. The correct way to do this is to use power series which we shall study in great detail in the next chapter. For the time being, we will assume that any non-zero complex number can be written in *polar form* as $z = |z|(\exp(i\theta)$ where θ is the angle that the line joining 0 to z makes with the x-axis and the function exp is realted to the familiar trigonometric functions by the famous Euler's formula:

$$\exp(i\theta) = \cos(\theta) + i\sin(\theta).$$

With the polar form in hand, it is easy to see that if z, w then the product has modulus the product of the moduli of z and w and makes an angle the sum of the angles z and w make with x-axis. We will make all this precise in Chapter 3.

1.3 The field structure on $\mathbb{C}$

As remarked before, $\mathbb{C}$ is a commutative field. It is natural to ask if it is ordered.

Definition 2. Let S be a set. A *total ordering* on S is a relation $\leq$ that satisfies

1. **Reflexivity**: $a \leq a$ for all a in S.

2. **Antisymmetry**: $a \leq b$ and $b \leq a$ implies $a = b$.
3. **Transitivity**: $a \leq b$ and $b \leq c$ implies $a \leq c$.
4. **Comparability** (trichotomy law): For any a, b in S, either $a \leq b$ or $b \leq a$.

If $a \leq b$ and $a \neq b$, we often write $a < b$.

Definition 3 (Ordered Field)**.** A field $\mathbb{K}$ with a total ordering $\leq$ is said to be an *ordered field* if it satisfies

- if $a \leq b$ then $a + c \leq a + c,\ \forall c \in \mathbb{K}$,
- if $0 \leq a$ and $0 \leq b$ then $0 \leq a \cdot b$.

Theorem 4. *$\mathbb{C}$ cannot be given the structure of an ordered field.*

Proof. Assume to the contrary that $\leq$ makes $\mathbb{C}$ into an ordered field. Then either $i > 0$ or $i < 0$. Suppose $i > 0$. Then $i^2 = -1 > 0$. Adding 1 to both sides, we see that $0 > 1$. On the other hand $-1 > 0$ implies that $(-1)(-1) = 1 > 0$. This is a contradiction. An analogous argument works for $i < 0$. □

Another natural question is the following: view $\mathbb{C}$ as sitting in $\mathbb{R}^3$ as the first two coordinates; Can we give $\mathbb{R}^3$ the structure of a commutative field so that $\mathbb{C}$ is a subfield?

Theorem 5. *$\mathbb{R}^3$ cannot be given the structure of a commutative field such that $\mathbb{C}$ is a subfield.*

Proof. $\mathbb{R}^3$ is obviously a vector space. Assume also that we have defined a multiplication $\cdot$ on $\mathbb{R}^3$ that makes it a commutative field that extends $\mathbb{C}$. Denote the basis vectors $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ by $1, i, j$, respectively. With this notation, $i^2 = -1$. Let us compute ij. Suppose $ij = a + bi + cj,\ a, b, c \in \mathbb{R}$. Observe first that

$$-j = (i^2)j = i(ij), \quad j^2 = -(ij)^2.$$

So,

$$-j = ai - b + c(ij) = ai - b + c(a + bi + cj).$$

Equating coefficients, we see that $c^2 = -1$ which is absurd. □

2 Linear mappings

In this section, we consider a $\mathbb{R}$-linear mapping $T : \mathbb{C} \to \mathbb{C}$, i.e., $T(\lambda_1 z_1 + \lambda_2 z_2) = \lambda_1 T(z_1) + \lambda_2 T(z_2),\ \forall \lambda_1, \lambda_2 \in \mathbb{R}$. Now, $\mathbb{C}$ is both a vector space over $\mathbb{R}$ of dimension 2 and vector space over $\mathbb{C}$ of dimension 1. It is natural to ask for conditions on an arbitrary map $T : \mathbb{C} \to \mathbb{C}$ that guarantee that T is $\mathbb{C}$-linear. Two obvious necessary conditions:

1. T is $\mathbb{R}$-linear,
2. T commutes with multiplication by i, i.e., $T(iz) = iT(z) \forall z \in \mathbb{C}$.

It turns out that these two conditions are sufficient to guarantee $\mathbb{C}$-linearity of T. Suppose T is $\mathbb{R}$-linear. Then for $z = x + iy$, $T(z) = xT(1) + yT(i)$. Writing $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$ and collecting terms we see that $T(z) = \alpha z + \beta \bar{z}$, where

$$\alpha = \frac{T(1) - iT(i)}{2}$$
$$, \beta = \frac{T(1) + iT(i)}{2}$$

If T is $\mathbb{C}$-linear, then $T(z) = zT(1)$ and therefore T is just multiplication by a scalar. From this, we see that an $\mathbb{R}$-linear map T is $\mathbb{C}$-linear iff the quantity β above is 0. This happens precisely when $T(i) = iT(1)$.

Writing $T(1) = (a, c)$ and $T(i) = (b, d)$, we see that the matrix of T under the standard basis is:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

If $T(i) = iT(1)$ then $a = d$ and $c = -b$. In matrix form, $\mathbb{C}$-linear maps have a matrix of type

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

A straighforward computation reveals that $\alpha = \frac{1}{2}(a + d - ib + ic)$ and $\beta = \frac{1}{2}(a - d + ic + ib)$. Another easy computation shows that

$$\det T = ad - bc = |\alpha|^2 - |\beta|^2,$$

which means that T is invertible iff $|\alpha| \neq |\beta|$. In this case, we can explicitly solve for the inverse T^{-1}:

$$T^{-1}(w) = \frac{\overline{\alpha} w - \beta \overline{w}}{|\alpha|^2 - |\beta|^2}.$$

Summarizing: there are ways to view a $\mathbb{R}$-linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$:

1. Maps that satisfy $T(x + iy) = xT(1) + yT(i)$. Such a map is $\mathbb{C}$-linear iff $T(i) = iT(1)$.
2. Maps that are of the form $T(z) = \alpha z + \beta \overline{z}$. Such a map is $\mathbb{C}$-linear iff $\beta = 0$.
3. As a matrix of the form
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
   In this representation, the map is $\mathbb{C}$-linear iff
$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$