MA5360 – Assignment 3
Due Date – NO DUE DATE!

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1. Show that the function $\cos z$ maps the strip $B = \{z : 0 < \mathrm{Re} z < \pi\}$ onto the domain $U = \mathbb{C} \setminus \{x \in \mathbb{R} : |x| \geq 1\}$ conformally and injectively. Find an expression for its inverse in terms of the logarithm.

2. In this exercise, you will work out the basics of complex powers. Given two complex numbers z, w with $z \neq 0$, we define the complex power denoted by $z^{[w]}$ to be the set

$$\{\exp wx : x \in \mathrm{Log} z\}.$$

   a) Briefly sketch a motivation for the above definition.
   b) What is $z^{[0]}$?
   c) Compute $i^{[i]}$.
   d) If w is an integer show that $z^{[w]}$ is a singleton set comprising of the complex number z^w.
   e) If $w = p/q$ is a rational show that the set z^w comprises the set of q-th roots of z^p.
   f) In all other cases show that z^w is a countably infinite set.
   g) Define $e := \exp(1)$ (which is real). Show that $\exp(z) \in e^{[z]}$ and thus we may henceforth use the notation e^z!
   h) Formulate and understand what it means for a function f to be a continuous branch of z^w.
   i) Is a continous branch of z^w automatically holomorphic?

3. If you have not already done so, write down an expression for the Laplacian in terms of the derivatives $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ derivatives.

4. Consider the function f defined on $\mathbb{R}$ by

$$f(x) := \begin{cases} \exp(-1/x^2) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

   Prove that $f \in C^\infty(\mathbb{R})$ but its Taylor series at the origin does not converge to f in any open interval containig 0.

5. Let f be holomorphic in a neighborhood of a closed rectangle R except for finitely many points $z_0, \ldots, z_n \in \mathrm{int}(R)$ and suppose that $\lim_{z \to z_j}(z - z_j)f(z) = 0$. Prove that $\int_R f(z)dz = 0$

6. Compute the integral

$$\int_0^{2\pi} e^{\cos\theta} \sin(n\theta - \sin\theta)d\theta$$

7. Prove that if f is a continuous function on an open convex set U and holomorphic on $U \setminus \{z_0\}, z_0 \in U$, then $\int_\gamma f(z)dz = 0$ for any closed contour $\gamma : [a, b] \to U$.

8. Let γ be a closed path in $\mathbb{C}$ that misses 0. Show directly that the value of

$$\frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-z_0}$$

is an integer.

9. Let $f \in \mathscr{O}(\mathbb{D}) \cap C^0(\overline{\mathbb{D}})$ and suppose vanishes on some arc on the unit circle. Conclude that $f \equiv 0$.

10. Let U be a bounded domain and suppose $f : U \to U$ is holomorphic and satisfies $f(a) = a$ and $f'(a) = 1$ for some $a \in U$. Show that f is linear.

11. Show that
$$\int_0^{\infty}\frac{\sin x}{x}dx = \frac{\pi}{2}.$$

12. Show that any two primitives F and G of a function $f \in \mathscr{O}(U)$ differ by a constant.

13. Let $L \subset \mathbb{C}$ be any line. Suppose $f \in \mathscr{O}(U \setminus L)$. Prove that $f \in \mathscr{O}(U)$.

14. Let f be an entire function with the property that for some polynomial $p(z)$, we have

$$|f(z)| \leq |p(z)|$$

for all $|z| > R, R \in \mathbb{R}$. What can you say about f?